BAIRE SPACES, TYCHONOFF POWERS AND THE VIETORIS TOPOLOGY*

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Abstract. In this paper, we show that if the Tychonoff power X^ω of a quasi-regular space X is Baire then its Vietoris hyperspace 2^X is also Baire. We provide two examples to show that the converse of this result does not hold in general, and the Baireness of finite powers of a space is insufficient to guarantee the Baireness of its hyperspace.

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1. Introduction

All topological spaces considered in this paper are assumed to be Hausdorff, although it is not always necessary to do so. Let X be a topological space. In this paper, 2^X denotes the hyperspace of X consisting of all nonempty closed subsets of X endowed with the Vietoris topology [7]. A canonical base for this topology is given by all subsets of 2^X of the form

$$\langle \mathscr{U} \rangle := \left\{ F \in 2^X : F \subseteq \bigcup \mathscr{U},\ F \cap V \neq \varnothing \text{ for any } V \in \mathscr{U} \right\},$$

where $\mathscr{U}$ runs over the finite families of nonempty open subsets of X. In the sequel, 2^X will always carry this topology. In addition, all powers of X are endowed with the Tychonoff product topology.

Recall that a space X is *Baire* if the intersection of every sequence of dense open subsets in X is dense. The well-known Baire category theorem claims that every complete metric or locally compact Hausdorff space is Baire. Due to the importance of the Baire category theorem in analysis and topology, it is interesting to know when the hyperspace 2^X of a Baire space X is still Baire. In 1975, McCoy [6] first considered this problem, and proved that if X is a quasi-regular Baire space with a countable pseudo-base, then 2^X is Baire. Here, a *pseudo-base* (also called a *π-base*) for X is a family of nonempty open sets $\mathscr{P}$ such that for every nonempty open set U of X there is $P \in \mathscr{P}$ with $P \subseteq U$. In the same paper, McCoy also indicated the interest in determing whether 2^X must be Baire for every metric Baire

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space X. Recently, this question of McCoy has been solved by Cao, García-Ferreira and Gutev in [1]. It is proved in [1] that if 2^X is Baire, then X^n is Baire for all $n < \omega$. By combining this result with a classical example in [3], it is concluded that there exists a metric Baire space X whose hyperspace 2^X is not Baire. Motivated by all these known facts, we shall consider the following question in this paper.

Question 1.1. *Given a space X, is there any relation between the Baireness of the countable power X^ω and that of 2^X?*

In Section 2, we first prove that if X is quasi-regular and X^ω is Baire, then 2^X is also Baire. As a corollary of this theorem, the main result of McCoy in [6] is deduced. Also, an affirmative solution of an oral question due to W. B. Moors is deduced from this theorem. In Section 3, we shall give two examples. The first one is a metric Baire space space X such that 2^X is Baire but X^ω is not Baire, and the second one is a metric Baire space all of whose finite powers are Baire but whose Vietoris hyperspace is not Baire. Our major tool in achieving these results is the following characterization of Baire spaces in terms of the Choquet game.

Theorem 1.2 ([4], [8], [10])**.** *A space X is Baire if, and only if the first player does not have a winning strategy in the Choquet game played in X.*

Recall that the *Choquet game* (or the *Banach-Mazur game*) $Ch(X)$ played in a space X is the following two-player infinite game. Players, called β (the first player) and α (the second player), alternatively choose nonempty open subsets of X with β starting first such that

$$U_0 \supseteq V_0 \supseteq U_1 \supseteq V_1 \supseteq \cdots .$$

In this way, a run $((U_n, V_n) : n < \omega)$ will be produced. Then α is said to *win* this run if $\bigcap_{n<\omega} U_n (= \bigcap_{n<\omega} V_n) \neq \varnothing$. Otherwise, we say that β has won. By a *strategy* σ for player β, we mean a function defined for all legal finite sequences of moves of α. If σ is a strategy for β in $Ch(X)$, $\sigma(\varnothing)$ denotes the first move of β. A finite sequence $(V_0, ..., V_n)$ of nonempty open sets of X is called a *partial play* of α subject to σ in $Ch(X)$ if $V_0 \subseteq \sigma(\varnothing)$ and $V_{i+1} \subseteq \sigma(V_0, ..., V_i) \subseteq V_i$ for all $i < n$. Similarly, an infinite sequence $(V_n : n < \omega)$ of nonempty open sets of X is called a *(full) play* of α subject to σ if $V_0 \subseteq \sigma(\varnothing)$ and $V_{n+1} \subseteq \sigma(V_0, ..., V_n) \subseteq V_n$ for all $n < \omega$. Strategies for player α, partial plays and (full) plays for β subject to a strategy of α can be defined similarly. In addition, a *winning strategy* for a player is a strategy such that this player wins each play of its opponent subject to this strategy no matter how the opponent moves in the game. If α has a winning strategy in $Ch(X)$, then X is called *weakly α-favorable*. Evidently, it follows from Theorem 1.2 that every weakly α-favorable space is Baire. However, Baire spaces which are not weakly α-favorable do exist, for example, any barely Baire space in [3] or any Bernstein set.

Readers should refer to [11] for more information on topological games.

2. The Baireness of X^ω implies that of 2^X

The main purpose of this section is to prove the following theorem which is stated in the title of the section.

Theorem 2.1. *Let X be a quasi-regular topological space. If X^ω is Baire, then 2^X is Baire.*

Recall that a space X is said to be *quasi-regular* if for every nonempty open set U of X there exists a nonempty open set V in X such that $\overline{V} \subseteq U$, where $\overline{V}$ is the closure of V in X. Theorem 2.1 has the following immediate consequences.

Corollary 2.2 ([6])**.** *If X is a quasi-regular Baire space having a countable pseudo-base, then 2^X is Baire.*

Proof. By Theorem 3 in [9], X^ω is Baire. Then the conclusion follows from Theorem 2.1. □

Corollary 2.3. *If X is a quasi-regular weakly α-favorable, then 2^X is Baire.*

Proof. It is known that the countable product of any family of weakly α-favorable spaces is weakly α-favorable. Thus, if X is weakly α-favorable, then X^ω is weakly α-favorable. Since every weakly α-favorable spaces is Baire, the conclusion follows from Theorem 2.1. □

The next corollary gives an affirmative answer to a recent oral question due to W. B. Moors.

Corollary 2.4. *If X is a metric hereditarily Baire space, then 2^X is Baire.*

Proof. It is shown in [2] that the product of any family of metric hereditarily Baire spaces is Baire. Thus, X^ω is Baire if X is a metric hereditarily Baire space. Now, the conclusion follows from Theorem 2.1. □

To prove Theorem 2.1, we need some preparation. Let $\tau_0(X)$ be the family of all nonempty open sets of a space X, and let $[\tau_0(X)]^{<\omega}$ be the family of all finite sets in $\tau_0(X)$. For each $\mathscr{U} = \{U_0, ..., U_{n-1}\} \in [\tau_0(X)]^{<\omega}$, let

$$[\mathscr{U}] = [U_0, ..., U_{n-1}] := \left(\prod_{i=0}^{n-1} U_i\right) \times X^{\omega \setminus n}$$

be the basic open set in X^ω defined by $\mathscr{U}$ in this particular order. Let $\mathscr{V} = \{V_0, ..., V_{m-1}\}$ be another member of $[\tau_0(X)]^{<\omega}$. Then, $[U_0, ..., U_{n-1}, \mathscr{V}]$ is defined by letting

$$[U_0, ..., U_{n-1}, \mathscr{V}] := \left(\prod_{i=0}^{n-1} U_i\right) \times \left(\prod_{i=0}^{m-1} V_i\right) \times X^{\omega \setminus (n \cup m)}.$$

If $[\mathscr{V}] \subseteq [\mathscr{U}]$, then we shall always assume $n \le m$ and $V_i \subseteq U_i$ for all $i < n$. Furthermore, if X is dense-in-itself, then it is easy to see that

$$\Re(X^\omega) := \{[\mathscr{U}] : \mathscr{U} \text{ is pairwise disjoint and } \mathscr{U} \in [\tau_0(X)]^{<\omega}\}$$

is a pseudo-base for X^ω. On the other hand, it can be verified that

$$\Im(2^X) := \{\langle \mathscr{U} \rangle : \mathscr{U} \text{ is pairwise disjoint and } \mathscr{U} \in [\tau_0(X)]^{<\omega}\}$$

is always a pseudo-base for 2^X for any space X. If $\langle \mathscr{V} \rangle$ and $\langle \mathscr{U} \rangle$ are members of $\Im(2^X)$, it can be verified that $\langle \mathscr{V} \rangle \subseteq \langle \mathscr{U} \rangle$ if, and only if for $i \leq n-1$ there is $j \leq m-1$ such that $V_j \subseteq U_i$ and $\bigcup \mathscr{V} \subseteq \bigcup \mathscr{U}$, see [7]. Thus, if $\langle \mathscr{V} \rangle \subseteq \langle \mathscr{U} \rangle$, then we can conclude $m \geq n$, and we shall always assume $V_i \subseteq U_i$ for all $i \leq n-1$ (after re-arranging terms).

Proof of Theorem 2.1. Suppose that σ is a strategy for player β in $Ch(2^X)$. We shall show that σ is not a winning strategy for β. To this end, we shall first apply σ to define inductively a strategy θ for β in $Ch(X^\omega)$, and then apply the Baireness of X^ω. Without loss of generality, we shall restrict moves of β and α in $Ch(2^X)$ on $\Im(2^X)$, since $\Im(2^X)$ is a pseudo-base for 2^X. First of all, we shall consider the case when X is dense-in-itself. In this case, $\Re(X^\omega)$ is a pseudo-base forX^ω. Thus, we shall restrict moves of β and α in $Ch(X^\omega)$ on $\Re(X^\omega)$.

Step 1. Suppose $\sigma(\varnothing) = \langle U_0^0, ..., U_{n_0-1}^0 \rangle$. Then, we define $\theta(\varnothing)$ by letting $\theta(\varnothing) := [U_0^0, ..., U_{n_0-1}^0]$. Put $n_{-1} = m_{-1} = 0$.

If α responds to $\theta(\varnothing)$ by selecting a finite family of nonempty open sets Π_0 such that $[\Pi_0] \in \Re(X^\omega)$ and $[\Pi_0] \subseteq \theta(\varnothing)$. Then, $|\Pi_0| \geq n_0$. For simplicity, we write $\Pi_0 = \Sigma_0^0 \cup \Gamma_0^0$, where $\Sigma_0^0 = \{V_0^0, ..., V_{n_0-1}^0\}$ and $\Gamma_0^0 = \{W_0^0, ..., W_{m_0-1}^0\}$. If $m_0 = 0$, $\Gamma_0^0 = \varnothing$. Further, since $V_i^0 \subseteq U_i^0$ for all $i < n_0$, then $\langle \Sigma_0^0 \rangle \subseteq \sigma(\varnothing)$. Suppose that $\sigma(\langle \Sigma_0^0 \rangle) = \langle U_0^1, ..., U_{n_1-1}^1 \rangle$, where $n_1 \geq n_0 + m_0$. Since X is quasi-regular, we may require

$$\overline{\sigma(\langle \Sigma_0^0 \rangle)} = \langle \overline{U_0^1}, ..., \overline{U_{n_1-1}^1} \rangle \subseteq \langle \Sigma_0^0 \rangle.$$

Evidently, we have $\overline{U_h^1} \subseteq U_h^0$ for each $h < n_0$. Define $\theta([\Pi_0])$ by letting

$$\theta([\Pi_0]) := [U_0^1, ..., U_{n_0-1}^1, \Gamma_0^0, U_{n_0}^1, ..., U_{n_1-1}^1].$$

Suppose that we have defined θ for all finite sequences $([\Pi_i] : i < k-1)$ of length $k-1$ $(k \geq 2)$ in $\Im(X^\omega)$ satisfying

(i) $\Pi_i = \bigcup_{j \leq i} (\Sigma_j^i \cup \Gamma_j^i)$ for all $i < k-1$;

(ii) $\Sigma_j^i = \{V_h^i : n_{j-1} \leq h < n_j\}$ and $\Gamma_j^i = \{W_h^i : m_{j-1} \leq h < m_j\}$ for all $j \leq i$ and $i < k-1$;

(iii) $n_0 \leq ... \leq n_{k-2}$ and $m_0 \leq ... \leq m_{k-2}$;

(iv) $n_i + m_i \leq n_{i+1}$ for all $i < k-2$;

(v) $[\Pi_{i+1}] \subseteq \theta([\Pi_0], ..., [\Pi_i]) \subseteq [\Pi_i]$ for all $i < k-2$;

(vi) $\sigma(\langle \Sigma_0^0 \rangle, ..., \langle \bigcup_{j \leq i} \Sigma_j^i \rangle) = \langle U_0^{i+1}, ..., U_{n_{i+1}-1}^{i+1} \rangle$ for all $i < k-1$;

(vii) $\langle \overline{U_0^{i+1}}, ..., \overline{U_{n_{i+1}-1}^{i+1}} \rangle \subseteq \langle \bigcup_{j \leq i} \Sigma_j^i \rangle$ for all $i < k-1$.

It follows from (vi) and (vii) that $\overline{U_h^{i+1}} \subseteq U_h^i$ for all $h < n_i$ and $i < k-1$.

Step $k+1$. Let $([\Pi_i] : i < k)$ be a finite sequence of length k in $\Im(2^X)$ such that $([\Pi_i] : i < k)$ satisfies (i)–(vii) and $[\Pi_{k-1}] = \bigcup_{j<k} (\Sigma_j^{k-1} \cup \Gamma_j^{k-1})$, where

$\Sigma_j^{k-1} = \{V_h^{k-1} : n_{j-1} \le h < n_j\}$ and $\Gamma_j^{k-1} = \{W_h^{k-1} : m_{j-1} \le h < m_j\}$ for all $j < k$. Suppose that

$$\sigma(\langle\Sigma_0^0\rangle, ..., \langle\bigcup_{i<k} \Sigma_i^{k-1}\rangle) = \langle U_0^k, ..., U_{n_k-1}^k\rangle$$

such that $\langle\overline{U_0^k}, ..., \overline{U_{n_k-1}^k}\rangle \subseteq \langle\bigcup_{i<k} \Sigma_i^{k-1}\rangle$, where $n_k \ge n_{k-1} + m_{k-1}$. Then, it is clear from the hypotheses that $\overline{U_h^{k+1}} \subseteq U_h^k$ for all $h < n_k$. Finally, we define $\theta([\Pi_0], ..., [\Pi_{k-1}])$ by letting

$$\theta([\Pi_0], ..., [\Pi_{k-1}]) := [U_0^k, ..., U_{n_0-1}^k, \Gamma_0^k, ..., U_{n_{k-2}}^k, ..., U_{n_{k-1}-1}^k, \Gamma_{k-1}^{k-1}, U_{n_{k-1}}^k, ..., U_{n_k-1}^k].$$

This completes the definition of the strategy θ.

Since X^ω is a Baire space, then θ is not a winning strategy for player β in $Ch(X^\omega)$. Hence, there exists a play $([\Pi_k] : k < \omega)$ of player α subject to θ in $Ch(X^\omega)$ such that $\bigcap_{k<\omega}[\Pi_k] \ne \varnothing$, where each $[\Pi_k]$ has the form as that in (i)–(vii). Note that $(\langle\bigcup_{j\le k} \Sigma_j^k\rangle : k < \omega)$ is a play for player α subject to σ in $Ch(2^X)$. Fix any arbitrary point $(z_i) \in \bigcap_{k<\omega}[\Pi_k]$. For each $k < \omega$, let $A_k = \{z_i : n_{k-1} + m_{k-1} \le i < n_k\}$. Then, put $A = \bigcup_{k<\omega} A_k$. It follows from the construction of θ that for each $i < \omega$,

$$\bigcup_{k\le i} A_k \in \sigma(\langle\Sigma_0^0\rangle, ..., \langle\bigcup_{j\le i-1} \Sigma_j^{i-1}\rangle).$$

Consequently, $\overline{A} \in \bigcap_{k<\omega}\langle\bigcup_{j\le k} \Sigma_j^k\rangle$. Therefore, σ is not a winning strategy for player β in $Ch(2^X)$. By Theorem 1.2, 2^X is a Baire space.

If X is not dense-in-itself, then some of open sets appeared in (i)–(vii) could be singletons. In this case, the previous argument still works with just a slight modification. □

3. Two examples

In this section, we shall present two examples as promised in the abstract. These examples are variations of the example given in Remark 9 of [3].

Let S^T be the collection of all functions from a set S to a set T. Given a cardinal κ, let FS_κ be the set of all finite sequences in κ, i.e., $FS_k = \bigcup\{\kappa^n : n < \omega\}$. For each $\sigma \in FS_\kappa$, let dom σ be the domain of σ. Then $\sigma^\frown\gamma$ is $\sigma \cup \{(\text{dom } \sigma, \gamma)\}$, that is, σ with γ stuck on the end. Let J_κ be the space of κ^ω equipped with the metric d defined such that for any $f,\ g \in \kappa^\omega$,

$$d(f,g) := \begin{cases} 0, & \text{if } f(n) = g(n) \text{ for all } n \in \omega; \\ 2^{-n}, & \text{if } f \ne g \text{ and } n \text{ is the least with } f(n) \ne g(n). \end{cases}$$

For any $f \in J_\kappa$, if cf$\kappa > \omega$, then we put $f^* := \sup\{f(n) + 1 : n < \omega\}$. A subset C of an infinite cardinal κ is called *cub* if it is closed unbounded, and a subset A of κ is called *stationary* in κ if A intersects every cub set C in κ. For basic properties of cub and stationary sets, readers should refer to [5]. Let $\mathfrak{c}$ be the continuum. The next cardinal after $\mathfrak{c}$ is $\mathfrak{c}^+$, and $C_\omega\mathfrak{c}^+$ is the subset of $\mathfrak{c}^+$ consisting of all ordinals of cofinality ω. It is known that $C_\omega \mathfrak{c}^+$ is stationary, and $C_\omega \mathfrak{c}^+$ can be split into $\mathfrak{c}^+$ disjoint stationary subsets of $\mathfrak{c}^+$. Let $\{A_x : x \in J_\omega\}$ be a family of disjoint stationary subsets of $C_\omega \mathfrak{c}^+$.

Example 3.1. *There exists a metric Baire space X such that 2^X is Baire, but X^ω is not Baire.* For each $y \in J_\omega$, define

$$C_y := \bigcup \{A_{y'} : y' \in J_\omega \text{ and } y(0) \neq y'(0)\},$$

and then our desired space X is defined by

$$X := \{\langle y, f\rangle \in J_\omega \times J_{\mathfrak{c}^+} : f^* \in C_y\}$$

and is equipped with the metric inherited from the product space $J_\omega \times J_{\mathfrak{c}^+}$. For each $\sigma \in FS_\omega$ and $\tau \in FS_{\mathfrak{c}^+}$, we shall define

$$B_{(\sigma,\tau)} := \{\langle y, f\rangle \in J_\omega \times J_{\mathfrak{c}^+} : y\restriction_{\text{dom } \sigma} = \sigma \text{ and } f\restriction_{\text{dom } \tau} = \tau\}.$$

- X^ω *is not Baire.*

To see this, for any $i, j, k < \omega$, let us define

$$D_{ijk} := \{\langle\langle y_0, f_0\rangle, \cdots\rangle \in X^\omega : \min(f_i^*, f_j^*) > \max(f_i(k), f_j(k))\}.$$

In addition, for each $\ell < \omega$, we define

$$E_\ell := \{\langle\langle y_0, f_0\rangle, \cdots\rangle \in X^\omega : \ell \subseteq \{y_0(0), ..., y_n(0)\} \text{ for some } n < \omega\}.$$

It can be checked all D_{ijk}'s and E_ℓ's are dense open in X^ω. We claim

$$\bigcap_{i,j,k,\ell<\omega}(D_{ijk} \cap E_\ell) = \varnothing.$$

If not, there exists a point $\langle\langle y_0, f_0\rangle, \cdots\rangle$ in X^ω such that

$$\langle\langle y_0, f_0\rangle, \cdots\rangle \in \bigcap_{i,j,k,\ell<\omega}(D_{ijk} \cap E_\ell).$$

Then, it is clear that $f_0^* = f_1^* = \cdots = \gamma$ for some $\gamma \in C_\omega \mathfrak{c}^+$. By definition, $\gamma \in C_{y_n}$ for all $n < \omega$. Thus, if we pick some $z \in J_\omega$ such that $\gamma \in A_z$, then $A_z \subseteq C_{y_n}$ for all $n < \omega$. This implies that $z(0) \neq y_n(0)$ for all $n < \omega$, and thus $z(0) \notin \{y_n(0) : n < \omega\}$. On the other hand, by definition of E_ℓ's, we have $\omega = \{y_n(0) : n < \omega\}$. This is a contradiction.

- 2^X *is Baire.*

Suppose that θ is a strategy for player β in $Ch(2^X)$. We shall show that θ is not a winning strategy for β. To this end, we need some preparation. Define $\mathscr{K}$ such that $\Sigma \in \mathscr{K}$ if and only if there are $m, n \in \omega$ such that $\Sigma = \{(\sigma_0, \tau_0), ..., (\sigma_{n-1}, \tau_{n-1})\}$, where $\sigma_i \in FS_\omega$ and $\tau_i \in FS_{\mathfrak{c}^+}$ for all $i < n$, and dom σ = dom $\tau = m$ for all $(\sigma, \tau) \in \Sigma$. Then,

$$\mathscr{B} := \{\langle\{B_{(\sigma,\tau)} \cap X : (\sigma, \tau) \in \Sigma\}\rangle : \Sigma \in \mathscr{K}\}$$

is a pseudo-base for 2^X. Assume that $\theta(\varnothing) \in \mathscr{B}$, and $(A_0, ..., A_{n-1})$ is a partial play for player α satisfying

$$A_0 \subseteq \theta(\varnothing),\ A_{k+1} \subseteq \theta(A_0, ..., A_k) \subseteq A_k$$

for all $k < n-1$ and $A_k \in \mathscr{B}$ for all $k < n$. For brevity, given $\Sigma \in \mathscr{K}$, put

$$\langle\Sigma\rangle := \langle\{B_{(\sigma,\tau)} \cap X : (\sigma,\tau) \in \Sigma\}\rangle.$$

We shall define a function (associated with θ) F_θ as follows:

(i) $\varnothing \in \text{dom } F_\theta$, and $F_\theta(\varnothing) = \Sigma$, where $\langle\Sigma\rangle = \theta(\varnothing)$;

(ii) for any $(\Sigma_0, ..., \Sigma_{n-1}) \in \mathscr{K}^n$, $n > 0$, if $(\langle\Sigma_0\rangle, ..., \langle\Sigma_{n-1}\rangle)$ is a partial play of player α subject to θ, then $(\Sigma_0, ..., \Sigma_{n-1}) \in \text{dom } F_\theta$ and, in this case, let $F_\theta(\Sigma_0, ..., \Sigma_{n-1}) = \Sigma$, where $\langle\Sigma\rangle = \theta(\langle\Sigma_0\rangle, ..., \langle\Sigma_{n-1}\rangle)$.

For any $\gamma < \mathfrak{c}^+$, define

$$\mathscr{K}\restriction_\gamma := \{\Sigma \in \mathscr{K} : \tau(i) \in \gamma \text{ for all } (\sigma,\tau) \in \Sigma \text{ and all } i \in \text{dom } \tau\}.$$

We shall call γ a *fixed point of* F_θ if $F_\theta[(\mathscr{K}\restriction_\gamma) \cap \text{dom } F_\theta] \subseteq \mathscr{K}\restriction_\gamma$. Let C be the set of all fixed points of F_θ. We claim that C is a cub set in $\mathfrak{c}^+$. First, we check that C is closed in $\mathfrak{c}^+$. To this end, let $\{\gamma_\xi : \xi < \mu\} \subseteq C$ and $\gamma = \sup\{\gamma_\xi : \xi < \mu\}$. Then

$$\begin{aligned} F_\theta[(\mathscr{K}\restriction_\gamma) \cap \text{dom } F_\theta] &= \textstyle\bigcup_{\xi<\mu} F_\theta[(\mathscr{K}\restriction_{\gamma_\xi}) \cap \text{dom } F_\theta] \\ &\subseteq \textstyle\bigcup_{\xi<\mu} \mathscr{K}\restriction_{\gamma_\xi} = \mathscr{K}\restriction_\gamma. \end{aligned}$$

To see that C is unbounded in $\mathfrak{c}^+$, let $\alpha < \mathfrak{c}^+$ be arbitrary. Define $\gamma_0 = \alpha$. Then we define γ_i $(i > 0)$ by induction such that

$$F_\theta[(\mathscr{K}\restriction_{\gamma_i}) \cap \text{dom } F_\theta] \subseteq \mathscr{K}\restriction_{\gamma_{i+1}}.$$

This is possible, since the size of the set

$$\{\tau(i) : \Sigma \in \mathscr{K}\restriction_{\gamma_i},\ (\sigma,\tau) \in \Sigma \text{ and } i \in \text{dom } \tau\}$$

is at most of $\mathfrak{c}$. Let $\gamma = \sup\{\gamma_i : i < \omega\}$. Then, it is easy to see that $\gamma \in C$.

Let $\Sigma_0 \neq \varnothing$ be such that $\langle\Sigma_0\rangle \subseteq \theta(\varnothing)$. Then, there is an $m > 0$ such that $\text{dom } \sigma = \text{dom } \tau = m$ for all $(\sigma,\tau) \in \Sigma_0$. Fix some $y' \in J_\omega$ such that $y'(0) \notin \{\sigma(0) : (\sigma,\tau) \in \Sigma_0\}$, and pick $\delta \in A_{y'} \cap C$ and an increasing sequence $\{\delta_i : i < \omega\}$ such that $\delta = \sup\{\delta_i : i < \omega\}$ and $\Sigma_0 \in \mathscr{K}\restriction_\delta$. By induction, let $\Sigma_0, ..., \Sigma_n$ be defined such that $(\langle\Sigma_0\rangle, ..., \langle\Sigma_n\rangle)$ is a partial play of α subject to θ and $\Sigma_i \in \mathscr{K}\restriction_\delta$ for all $i \leq n$. Put $\Sigma = F_\theta(\Sigma_0, ..., \Sigma_n)$, and then define

$$\Sigma_{n+1} := \{(\sigma^\frown 0, \tau^\frown \delta_n) : (\sigma,\tau) \in \Sigma\} \in \mathscr{K}.$$

Then $(\langle\Sigma_0\rangle, ..., \langle\Sigma_{n+1}\rangle)$ is a partial play of α subject to θ, and

$$(\Sigma_0, ..., \Sigma_{n+1}) \in \mathscr{K}\restriction_\delta.$$

Continue this process inductively, we produce a play $\{\langle\Sigma_n\rangle : n < \omega\}$ of player α subject to θ. We claim $\bigcap_{n<\omega}\langle\Sigma_n\rangle \neq \varnothing$. To see this, let

$$F := \textstyle\bigcap_{n<\omega} \bigcup\{B_{(\sigma,\tau)} : (\sigma,\tau) \in \Sigma_n\} \qquad (*)$$

It is clear that F is a closed set in $J_\omega \times J_{\mathfrak{c}^+}$. In the sequel, we shall verify that $F \in \bigcap_{n<\omega}\langle\Sigma_n\rangle$. First, we show that $F \subseteq X$. Take any $\langle y, f\rangle \in F$. There is a sequence $\{m_n : n < \omega\}$ such that for every $(\sigma,\tau) \in \Sigma_n$, $\text{dom } \sigma = \text{dom } \tau = m_n$. It follows that $(y\restriction_{m_n}, f\restriction_{m_n}) \in \Sigma_n$ for all $n < \omega$. Thus, $f^* = \delta \in A_{y'}$.

Since $y'(0) \neq y(0)$, then $A_{y'} \subseteq C_y$. This implies that $\langle y, f\rangle \in X$. Second, we show $B_{(\sigma,\tau)} \cap F \neq \varnothing$ for any $n \in \omega$ and for any $(\sigma, \tau) \in \Sigma_n$. Since $\langle \Sigma_{n+k+1}\rangle \subseteq \langle \Sigma_{n+k}\rangle$ for every $k < \omega$, by induction over $k < \omega$, there are $(\sigma_n, \tau_n) = (\sigma, \tau)$ and $(\sigma_{n+k}, \tau_{n+k}) \in \Sigma_{n+k}$ such that $(\sigma_{n+k}, \tau_{n+k})$ extends $(\sigma_{n+k'}, \tau_{n+k'})$ whenever $k > k'$. Then, it follows that

$$(\bigcup_{k<\omega} \sigma_{n+k},\ \bigcup_{k<\omega} \tau_{n+k}) \in \bigcap_{k<\omega} B_{(\sigma_{n+k},\tau_{n+k})} \subseteq F.$$

This implies $F \cap B_{(\sigma,\tau)} \neq \varnothing$. Thus, we have verified $F \in \bigcap_{n<\omega}\langle \Sigma_n\rangle$.

The argument in the previous paragraph shows that $\{\langle \Sigma_n\rangle : n < \omega\}$ is a play which witnesses θ not to be a winning strategy for player β in $Ch(2^X)$, and thus by Theorem 1.2, 2^X is Baire. □

The proof of the next lemma is similar to that of Lemma 1 in [3].

Lemma 3.2. *Let $\kappa > \omega$ be a regular cardinal. If $K \subseteq (J_\kappa)^m$ is closed and*

$$\{\alpha : f_0^* = \cdots = f_{m-1}^* = \alpha \text{ and } \langle f_0, ..., f_{m-1}\rangle \in K\}$$

is stationary, then there is a cub set C in κ such that $C \cap C_\omega\kappa \subseteq W$.

Example 3.3. *There exists a metric space X such that X^n is Baire for all $n < \omega$, but 2^X is not Baire.* For each $y \in J_\omega$, define a subset $I_y \subseteq J_\omega$ such that $y' \in I_y$ if, and only if

$$\max\{n \in \omega : y(n) = y'(n)\} \leq \max\{n \in \omega : y'(n) \geq n\} < \omega.$$

Then, let $C_y := \bigcup\{A_{y'} : y' \in I_y\}$, and our desired space X is defined by

$$X := \{\langle y, f\rangle \in J_\omega \times J_{\mathfrak{c}^+} : f^* \in C_y\},$$

and is equipped with the metric inherited from the product space $J_\omega \times J_{\mathfrak{c}^+}$.

- *X^m is Baire for all $m < \omega$.*

To do this, fix an $m < \omega$, and let $\mathscr{D} = \{D_i : i < \omega\}$ be a family of dense open sets in $(J_\omega \times J_{\mathfrak{c}^+})^m$, V a nonempty open subset in $(J_\omega \times J_{\mathfrak{c}^+})^m$. Put

$$W := \{\alpha < \mathfrak{c}^+ : f_0^* = \cdots = f_{m-1}^* = \alpha \text{ and } \langle\langle y_0, f_0\rangle, ..., \langle y_{m-1}, f_{m-1}\rangle\rangle \in V \cap (\bigcap_{i<\omega} D_i)\}.$$

We claim that W is stationary. Let C be a cub set in $\mathfrak{c}^+$. Define inductively a decreasing family of clopen subsets $\{H_n : n < \omega\}$ in $(J_\omega \times J_{\mathfrak{c}^+})^m$ whose diameter converges to 0 such that $H_0 \subseteq V$, $H_{2\ell+2} \subseteq D_\ell$, and $H_{2\ell+1}$ insures $f_0^* = \cdots = f_{m-1}^* \in C$, where

$$\{\langle\langle y_0, f_0\rangle, ..., \langle y_{m-1}, f_{m-1}\rangle\rangle\} = \bigcap_{i<\omega} H_i.$$

This verifies the claim.

For a point $p = \langle\langle y_0, f_0\rangle, ..., \langle y_{m-1}, f_{m-1}\rangle\rangle$ in $(J_\omega \times J_{\mathfrak{c}^+})^m$, an $h \in \omega^\omega$ and $i \in \omega$, let $B(p, 2^{-h(i)})$ be the ball centered at p with radius $2^{-h(i)}$, i.e., $\langle\langle \overline{y}_0, \overline{f}_0\rangle, ..., \langle \overline{y}_{m-1}, \overline{f}_{m-1}\rangle\rangle \in B(p, 2^{-h(i)})$ if, and only if $\overline{y}_j \restriction_{h(i)} = y_j \restriction_{h(i)}$ and $\overline{f}_j \restriction_{h(i)} = f_j \restriction_{h(i)}$ for all $j < m$. For each $\vec{y} = \langle y_0, ..., y_{m-1}\rangle \in (J_\omega)^\omega$, define

$$W_{\vec{y}h} := \{\alpha : f_0^* = \cdots = f_{m-1}^* = \alpha \text{ and } B(p, 2^{-h(i)}) \subseteq D_i \cap V \text{ for all } i \in \omega\}.$$

Since $W \subseteq \bigcup\{W_{\vec{y}h} : \vec{y} \in J_\omega, h \in \omega^\omega\}$ and W is not the union of $\mathfrak{c}$ non-stationary sets, it follows that there are $h \in \omega^\omega$ and

$$\vec{y} = \langle y_0, ..., y_{m-1}\rangle \in (J_\omega)^m$$

such that $W_{\vec{y}h}$ is stationary. By Lemma 3.2, there is a cub set C in $\mathfrak{c}^+$ with $C \cap C_\omega\mathfrak{c}^+ \subseteq W_{\vec{y}h}$. Choose $\hat{y} \in (m+1)^\omega$ with $\hat{y}(n) \notin \{y_0(n), \cdots y_{m-1}(n)\}$ for all $n \in \omega$. By definition, $A_{\hat{y}} \subseteq C_{y_i}$ for all $i < \omega$. Let

$$\beta \in A_{\hat{y}} \cap C \subseteq A_{\hat{y}} \cap W_{\vec{y}h}.$$

Then there exists $\langle f_0, ..., f_{m-1}\rangle \in (J_{\mathfrak{c}^+})^m$ such that

$$\langle\langle y_0, f_0\rangle, ..., \langle y_{m-1}, f_{m-1}\rangle\rangle \subseteq V \cap (\textstyle\bigcap_{i\in\omega} D_i)$$

and $f_0^* = \cdots = f_{m-1}^* = \beta$. Thus, $f_k^* \in A_{\hat{y}} \subseteq C_{y_k}$ for all $k \in \omega$. Therefore,

$$V \cap (\textstyle\bigcap_{i<\omega} D_i) \cap X^m \neq \varnothing,$$

which implies that $(\bigcap_{i<\omega} D_i) \cap X^m$ is dense in X^m, and thus X^m is Baire.

• 2^X *is not Baire.*

We shall define inductively a winning strategy θ for player β in $Ch(2^X)$. To this purpose, let $\mathscr{K}$ and $\mathscr{B}$ be defined as in Example 3.1. Given $\Sigma \in \mathscr{K}$, let $m \in \omega$ be such that dom σ = dom $\tau = m$ for all $(\sigma, \tau) \in \Sigma$. Define

$$\delta_\Sigma := \max\{\tau(k) : (\sigma,\tau) \in \Sigma, k \leq m\} + 1,$$

$$F(\Sigma) := \{(\sigma^\frown k, \tau^\frown \delta_\Sigma) : (\sigma,\tau) \in \Sigma,\ k < m\}.$$

Let $\theta(\varnothing) \in \mathscr{B}$ be arbitrary. Suppose that $(\langle\Sigma_0\rangle, ..., \langle\Sigma_{n-1}\rangle)$ is a partial play of α subject to θ in $Ch(2^X)$. Then, we define $\theta(\langle\Sigma_0\rangle, ..., \langle\Sigma_{n-1}\rangle) := \langle F(\Sigma_{n-1})\rangle$. Continuing this process inductively, we shall define the strategy θ. To see that θ is winning strategy for β, we will verify $\bigcap_{n<\omega}\langle\Sigma_n\rangle = \varnothing$ for any play $\{\langle\Sigma_n\rangle : n \in \omega\}$ of α subject to θ. In case that

$$Y := X \cap \textstyle\bigcap_{n<\omega}(\bigcup\{B_{(\sigma,\tau)} : (\sigma,\tau) \in \Sigma_n\}) = \varnothing,$$

we are done. Otherwise, one can pick $\langle y, f\rangle \in Y$. Since $\langle y, f\rangle \in X$, there exists $y' \in J_\omega$ such that $f^* \in A_{y'}$ and $\{n \in \omega : y'(n) \geq n\}$ is finite. Let

$$M := \max\{n \in \omega : y'(n) \geq n\}.$$

Let $N \in \omega$ be such that $m_N > M$ and $(y \restriction_{m_N},\ f \restriction_{m_N}) \in \Sigma_N$. By hypothesis,

$$((y \restriction_{m_N})^\frown k,\ (f \restriction_{m_N})^\frown \delta_{\Sigma_N}) \in F(\Sigma_N)$$

for each $k < m_N$. Since $m_N > M$, it follows that $y'(m_N) < m_N$. Therefore,

$$((y \restriction_{m_N})^\frown y'(m_N),\ (f \restriction_{m_N})^\frown \delta_{\Sigma_N}) \in F(\Sigma_N).$$

Since $\langle\Sigma_{N+1}\rangle \subseteq \langle F(\Sigma_N)\rangle$, $((y \restriction_{m_N})^\frown y'(m_N), (f \restriction_{m_N})^\frown \delta_{\Sigma_N})$ has an extension $(\overline{\sigma}, \overline{\tau}) \in \Sigma_{N+1}$.

Suppose that F is a nonempty subset of X such that $F \in \bigcap_{n\in\omega}\langle\Sigma_n\rangle$. In particular, $F \in \langle\Sigma_{N+1}\rangle$ and $F \cap B_{(\overline{\sigma},\overline{\tau})} \neq \varnothing$. So, there is $\langle x, g\rangle \in F \cap B_{(\overline{\sigma},\overline{\tau})}$. Let $\{m_n : n \in \omega\}$ be such that dom σ = dom $\tau = m_n$ for all $(\sigma, \tau) \in \Sigma_n$. Then $(x \restriction_{m_n},\ g \restriction_{m_n}) \in \Sigma_n$ and $(y \restriction_{m_n},\ f \restriction_{m_n}) \in \Sigma_n$ for all $n < \omega$. Thus,

$$\begin{aligned} g(m_n) = f(m_n) = \delta_{\Sigma_n} &= \max\{g(k) : k < m_n\} \\ &= \max\{f(k) : k < m_n\}. \end{aligned}$$

Therefore, $g^* = f^* \in A_{y'} \subseteq C_x$. This implies that $y' \in I_x$, and thus

$$M \geq \max\{n \in \omega : x(n) = y'(n)\}.$$

However, $m_N > M$ and $x(m_N) = \overline{\sigma}(m_N) = y'(m_N)$. This is a contradiction. Therefore, θ is a winning strategy for player β. □

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