Pricing Variance Swaps under Stochastic Volatility Model with Regime Switching - Discrete Observations Case

Guang-Hua Lian*
Collaboration with Robert Elliott
University of Adelaide

Feb. 2, 2011

Outline

- **Background**
- **An analytical solution for pricing variance swaps based on the Heston (1993) stochastic volatility model with regime switching**
- **Examples and Discussions**
- **Concluding Remarks**

Financial Models

- The first generation model: Black-Scholes model

$$dS = rSdt + \sigma SdB_t$$

Black-Scholes formula

$$C_t = S_t N(d_1) - K \exp\left[-r(T-t)\right] N(d_2)$$

where

$$d_1 = \frac{\ln S_t/K + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}}$$

$$d_2 = d_1 - \sigma\sqrt{T-t}$$

- It is incapable of generating "volatility smile".

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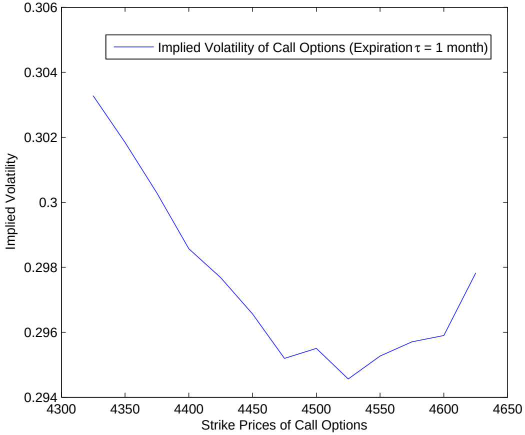
Financial Models

The implied volatility is calculated from the ASX/SPI200 index call options which will expire in one month. Data are obtained from Australia Stock Exchange, on Feb. 8, 2010. The ASX/SPI index is 4521 on that date.



The second generation of models.

- Stochastic volatility models (Heston 1993; Stein and Stein 1991)
- Jump diffusion models (Bakshi et al. 1997; Duffie et al. 2000)
- Local volatility surface models (Dupire B. 1994).

The third generation of models.

- Models incorporating regime switching.
- Levy jump models (CGMY);
- VG models;

Why Regime Switching?

- Economic reasons: business cycles.
- It is necessary to allow the key parameters of the model to respond to the general market movements.

Why Regime Switching?

- Empirical evidence: variation in parameters, e.g. Brown Dybvig (1986) and Gibbons Ramaswany (1993).
- Vo (2009) found strong evidence of regime-switching in the market, and showed that the regime-switching stochastic volatility model does a better job in capturing major events affecting the market.

Regime Switching Model in Finance Research

The applications of regime switching models in finance include

- asset allocation (Elliott & Van der Hoek 1997);
- short term rate model and bond evaluation (Elliott & Siu 2009);
- portfolio analysis (Zhou & Yin 2004; Honda 2003);
- pricing options (Guo & Zhang 2004);
- risk management (Elliott et al. 2008).

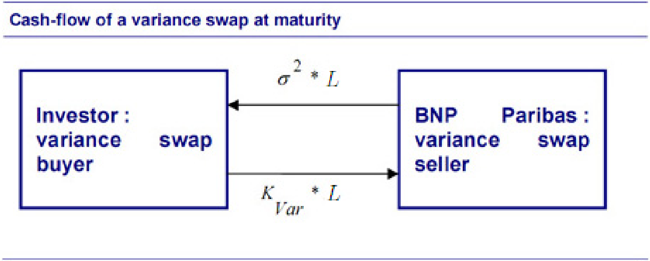
There is a little work on pricing variance swaps in the context of regime-switching models.

- The only paper so far is Elliott et al. (2007).
- Their work for variance swaps is based on continuous observations in calculating realized variance.
- They have also pointed out that in practice, variance swaps are always written on the realized variance evaluated by a discrete summation based on daily closing prices, instead of a continuous observations.

- **What is a variance swap?**

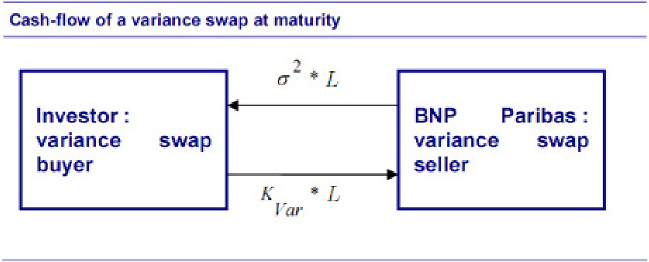
A variance swap is a forward contract on the **future** realized variance of the underlying asset.

- Cash flow of a variance swap at expiration



i) the σ_R^2 is the annualized realized variance over the contract life T;
ii) K_{var} is the annualized strike price for the variance swap.

Background



- The payoff of a variance swap at maturity T is usually of the form:

$$V_T = (\sigma_R^2 - K_{var}) \times L,$$

and L is the notional amount of the swap per annualized volatility point squared, which is usually set to 10000.

Background

- **There are several different forms of σ_R^2:**

$$\sigma_R^2 = \frac{AF}{N} \sum_{k=1}^{N} \left(\frac{S_{t_k} - S_{t_{k-1}}}{S_{t_{k-1}}}\right)^2 \tag{1}$$

or

$$\sigma_R^2 = \frac{AF}{N} \sum_{k=1}^{N} [Ln(S_{t_k}) - Ln(S_{t_{k-1}})] \tag{2}$$

or

$$\sigma_R^2 = \frac{1}{T} \int_0^T v_t dt \tag{3}$$

- **Analytical Approaches:**

  – Carr and Madan (1998), Demeterfi et al. (1999): replicate a variance swap by a portfolio of options;

  – Heston (2000): analytical solution based on GARCH model;

  – Howison (2004): continuously-sampled variance swaps based on stochastic volatility.

The limitation of these methods is the assumption that sampling frequency is high enough to allow the realized variance to be approximated by a continuously-sampled variance defined as

$$\sigma_R^2 = \frac{1}{T}\int_0^T v_t dt \tag{4}$$

- **Numerical Approaches:**

  – Little and Pant (2001): Finite difference method for discretely-sampled realized variance;

  – Windcliff et al. (2006): Integral differential equation approach for discretely sampled realized variance;

The drawback of these numerical approaches is that they are limited to the case with local volatility being a given function of the underlying asset and time.

- **Most Recent Research:**

  To properly address the discretely sampling effect, several works have been completed, based on the Heston stochastic volatility model (SV)

  – Broadie & Jain (2008);

  – Itkin & Carr (2010);

  – Zhu & Lian (2010);

The contributions of this study

Models	Continuous sampling case	Discrete sampling case
SV	Many	Zhu & Lian (2010)
SV with regime switching	Elliott et al. (2007)	No exact formula

Our Closed-form Analytical Solution

- **Assumptions:**
  – Consider a continuous-time finite-state Markov chain $X = \{X_t\}_{t \in T}$

$$X_t = X_0 + \int_0^t AX_s ds + M_t, \tag{5}$$

where M_t is an martingale.
The finite-state space is identified with $S = \{e_1, e_2, ..., e_N\}$, where $e_i = (0, ..., 1, ..., 0) \in R^N$

Our Closed-form Analytical Solution

- **Assumptions:**
  – The realized variance is discretely sampled and defined as

$$\sigma_R^2 = \frac{AF}{N} \sum_{i=1}^{N} \left(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}}\right)^2 \tag{6}$$

  – The underlying asset and the instantaneous variance follow the dynamics:

$$\begin{aligned} dS_t &= r_t S_t dt + \sqrt{V_t} S_t dB_t^S, \\ dV_t &= \kappa(\theta_t - V_t)dt + \sigma_V \sqrt{V_t} dB_t^V, \end{aligned} \tag{7}$$

respectively.

Our Closed-form Analytical Solution

$$dS_t = r_t S_t dt + \sqrt{V_t} S_t dB_t^S,$$
$$dV_t = \kappa(\theta_t - V_t)dt + \sigma_V \sqrt{V_t} dB_t^V.$$

- Here r is the risk-free interest rate, θ is the long-term mean of the variance, κ is a mean-reverting speed parameter of the variance, σ_V is the so-called volatility of volatility.

$$r_t = r(t, X_t) = < r, X_t >, \quad r = (r_1, r_2, ..., r_N)$$
$$\theta_t = \theta(t, X_t) = < \theta, X_t >, \quad \theta = (\theta_1, \theta_2, ..., \theta_N)$$

- dB_t^S and dB_t^V are two Wiener processes that are correlated by a constant correlation coefficient ρ, that is $< B_t^S, B_t^V >= \rho t$.

Our Closed-form Analytical Solution

Clearly, to calculate the price of an existing variance swap with a payoff $V_T = (\sigma_R^2 - K_{var}) \times L$ or to set up a strike price K_{var} for a new contract, essentially, all one needs is to calculate the expectation of the unrealized variance:

$$K_{var} = E_0^Q[\sigma_R^2] = E_0^Q[\frac{1}{T}\sum_{i=1}^{N}(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2],$$

where E_t^Q denotes the expectation under the Q measure conditional on the information available at time t.

Our Closed-form Analytical Solution

If we further assume that the sampling points are equally spaced, i.e.,

$$AF = \frac{1}{\Delta t} = \frac{N}{T},$$

then

$$K_{var} = E_0^Q[\sigma_R^2] = E_0^Q[\frac{1}{N\Delta t}\sum_{i=1}^{N}(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2].$$

Thus, our problem essentially becomes to evaluate N expectations

$$E_0^Q[(\frac{S_{t_i} - S_{t_{i-1}}}{S_{t_{i-1}}})^2] \tag{8}$$

Our Closed-form Analytical Solution

Characteristic Function Method:

- Assuming the current time is t, write $y_T = \log S_{T+\Delta} - \log S_T$.
- Define forward characteristic function $f(\phi; t, T, \Delta, V_t)$ of the stochastic variable y_T as the Fourier transform of the probability density function of y_T, i.e.,

$$f(\phi; t, T, \Delta, V_t) = E_t^{\mathbb{Q}}[e^{\phi y_T}]$$

$$= \mathsf{E}_t^{\mathbb{Q}}[\exp(\phi(\log S_{T+\Delta} - \log S_T))]$$

- Obtain this characteristic function and then solve the pricing of variance swaps.

Our Closed-form Analytical Solution

We combine the techniques of the tower rule (law of iterated expectation) and the partial differential equation (PDE).

- Step 1: conditional expectation.
  Given the filtration $F^X_{T+\Delta}$, the parameters r_t and θ_t can be considered to be time-dependent deterministic functions.

- Step 2: characteristic function of regime switching process, X_t;
  Solve the PDE associated with the regime switching process;

- Step 3: unconditional expectation;
  Apply the results in step 1 and 2 to finally obtain the required characteristic function.

... mathematical derivations ...

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Our Closed-form Analytical Solution

Proposition 0.1

If the underlying asset follows the dynamics (7), then the forward characteristic function of the stochastic variable $y_T = \log S_{T+\Delta} - \log S_T$ *is given by:*

$$f(\phi; t, T, \Delta, V_t) = E_t^{\mathbb{Q}}[e^{\phi y_T}] \tag{9}$$

$$= \exp\left(G(D(\phi, T), T - t)V_t\right) < \Phi(t, T)X_t, I > \tag{10}$$

where $D(\phi, t)$ *is given by,*

$$\begin{cases} D(\phi, t) = \dfrac{a+b}{\sigma_V^2} \dfrac{1 - e^{b(T+\Delta-t)}}{1 - g e^{b(T+\Delta-t)}} \\ a = \kappa - \rho\sigma_V\phi, \quad b = \sqrt{a^2 + \sigma_V^2(\phi - \phi^2)}, \quad g = \dfrac{a+b}{a-b} \end{cases} \tag{11}$$

Proposition 0.2

(Continue)
If the underlying asset follows the dynamics (7), then the forward characteristic function of the stochastic variable $y_T = \log S_{T+\Delta} - \log S_T$ *is given by:*

$$f(\phi; t, T, \Delta, V_t) = E_t^{\mathbb{Q}}[e^{\phi y_T}] \quad (12)$$
$$= \exp\left(G(D(\phi, T), T - t)V_t\right) < \Phi(t, T)X_t, I > \quad (13)$$

where $G(\phi; t, T, V_t)$ *is given by,*

$$\begin{cases} G(\phi, t) = \dfrac{2\kappa\phi}{\sigma_V^2\phi + (2\kappa - \sigma_V^2\phi)e^{\kappa(T-t)}} \\ J(t) = (1 - H_T(t))(\kappa\theta G(D(\phi, T), t)) + H_T(t)(r\phi + \kappa\theta D(\phi, t)) \\ \Phi(t, T) = \exp\left(\displaystyle\int_t^{T+\Delta} A' + diag(J(s))ds\right) \end{cases} \quad (14)$$

Our Closed-form Analytical Solution

- Having worked out the forward characteristic function

$$f(\phi; t, T, \Delta, V_t) = E_t^{\mathbb{Q}}[e^{\phi y_T}]$$

- Pricing variance swaps becomes quite trivial.

$$K_{var} = \frac{1}{T}\sum_{k=1}^{N}[f(2; 0, t_{k-1}, \Delta t, V_0) - 2f(1; 0, t_{k-1}, \Delta t, V_0) + 1]$$

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Numerical Results

- Obtain numerical results from the implementation of our pricing formula.
- Monte Carlo benchmark values for testing purpose.
- Compare with the continuous sampling approximation.

Numerical Results

- The model

$$
\begin{aligned}
dS_t &= rS_t dt + \sqrt{V_t} S_t dB_t^S, \\
dV_t &= \kappa(\theta - V_t)dt + \sigma_V \sqrt{V_t} dB_t^V, \\
&< B_t^S, B_t^S >= \rho t
\end{aligned}
$$

$$
\begin{aligned}
r_t &= r(t, X_t) =< r, X_t >, \quad r = (r_1, r_2, ..., r_N) \\
\theta_t &= \theta(t, X_t) =< \theta, X_t >, \quad \theta = (\theta_1, \theta_2, ..., \theta_N) \\
X_t &= X_0 + \int_0^t AX_s ds + M_t,
\end{aligned}
$$

- Parameters $\rho = -0.82$; $\kappa = 3.46$;
  $\sigma_V = 0.14$; $V_0 = (8.7/100)^2$;
  $A = [-0.1, 0.1; 0.4, -0.4]$; $X_0 = 1$;
  $r = [0.06; 0.03]$; $\theta = [0.009; 0.004]$.

Semi-Monte Carlo Simulations

- MC simulations are frequently used, particularly when no closed-form solutions.
- obtain benchmark values for testing other methods.
- not feasible for practical use because of computational inefficiency.

Semi-Monte Carlo Simulations

- We suggest a semi-MC method
- Algorithm.

1. Let N be the number of samplings. For each $n = 1, ..., N$, we then
2. obtain the n-th sampling path of the regime switching process, X_T;
3. with a realized sampling path of X_T, the characteristic function is presented in Proposition 1.

$$f(\phi; t, T, \Delta, V_t | F^X_{T+\Delta}) = E^{\mathbb{Q}}[e^{\phi y_T} | F^S_t \vee F^V_t \vee F^X_{T+\Delta}]$$
$$= e^{C(\phi,T)} g(D(\phi, T); t, T, V_t)$$

So we can calculate the price of a variance swap for the n-th sampling path.

4. calculate the average $K = \frac{1}{N} \sum_{n=1}^{N} K_n$.

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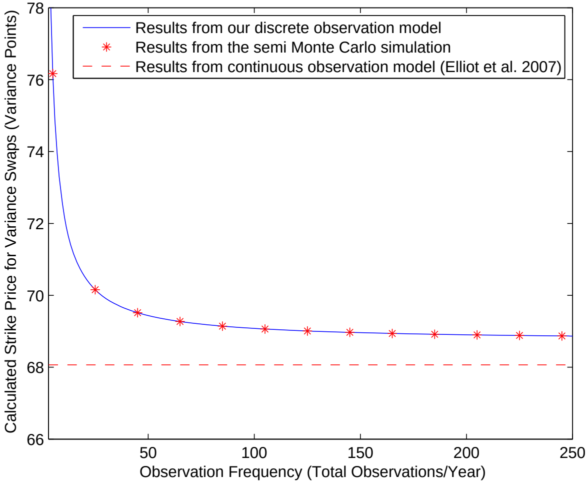
The continuous observation case

- Elliott et al. (2007)'s formula

$$P(X)=e^{-\int_0^T r_u\,du}N\left[\frac{\sigma_0^2}{\beta T}\left(1-e^{-\beta T}\right)+\frac{\beta}{T}\int_0^T\left(\int_0^t\langle\tilde{\alpha}^2,X_t\rangle e^{-\beta(t-s)}\,ds\right)dt-K_v\right]$$

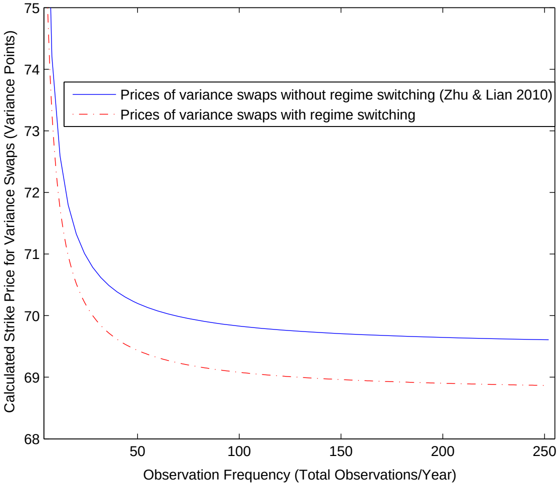
Results and Discussions

- A comparison with the results obtained from other approaches:



Results and Discussions

- A comparison with the results obtained from other approaches:



Concluding Remarks

- **An analytical solution is obtained for variance swaps based on a stochastic volatility model with regime switching;**
- **For discretely sampled variances, it is more accurate to use our solution than using continuous approximations;**
- **It examines the effect of ignoring regime switching on pricing variance and volatility swaps;**
- **Our solution can be very efficiently computed; substantial computational time can be saved in comparison to Monte Carlos Method;**

Thank you!

Author: Guanghua Lian*
Address: School Mathematical Science,
The University of Adelaide
Email: guanghua.lian@adelaide.edu.au