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Counter-Examples and Paradoxes in Teaching Mathematical Statistics: A Case Study

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Introduction and Framework

Counter-examples are a powerful and effective tool for scientists, researchers and practitioners. They are good indicators showing that a suggested hypothesis or a chosen direction of research is wrong. Before trying to prove a conjecture or a hypothesis it is often worth to look for a possible counter-example. It can save lots of time and effort. Creating examples and counter-examples is neither algorithmic nor procedural and requires advanced thinking which is not often taught at school (Selden & Selden, 1998; Tall, 1991; Tall, et al. 2001). Many students are used to concentrate on techniques, manipulations, familiar procedures and do not pay much attention to concepts, conditions of theorems and rules, reasoning and justifications. As Seldens argue, coming up with examples requires different cognitive skills from carrying out algorithms – one needs to look at mathematical objects in terms of their properties. To be asked for an example can be disconcerting. Students have no prelearned algorithms to show the 'correct way' (Selden & Selden, 1998).

There are several publications on using counter-examples in teaching/learning of mathematics, in particular calculus (Gelbaum and Olmstead, 1964; Peled & Zaslavski, 1997; Zaslavski & Ron, 1998; Bermudez, 2004; Gruenwald & Klymchuk, 2003; Klymchuk, 2004 & 2005). There are three well-known books on counter-examples in statistics at an advanced level (Stoyanov, 1997; Romano, 1986; Wise & Hall, 1993). But we could not find any publication on *using* counter-examples in teaching/learning of *first-year* probability and statistics. So we decided to apply counter-examples along with paradoxes as a pedagogical strategy in our first-year probability course.

The main objective of the study was to check our assumptions on how effective the usage of counter-examples is for deeper conceptual understanding, eliminating students' misconceptions and developing creative learning environment in teaching/learning of first-year probability course.

In this study, practice was selected as the basis for the research framework and, it was decided 'to follow conventional wisdom as understood by the people who are stakeholders in the practice' (Zevenbergen & Begg, 1999). The theoretical framework was based on Piaget's notion of cognitive conflict (Piaget, 1985). Some studies in mathematics education at secondary level (Swan, 1993; Irwin, 1997) found conflict to be more effective than direct instruction. 'Provoking cognitive conflict to help students understand areas of mathematics is often recommended' (Irwin, 1997). Swedosh and Clark (1997) used conflict in their intervention method to help undergraduate students to eliminate their misconceptions. 'The method essentially involved *showing* examples for which the misconception could be seen to lead to a ridiculous conclusion, and, having established a conflict in the minds of the students, the correct concept was taught' (Swedosh and Clark, 1997). Another study by (Horiguchi & Hirashima, 2001) used a similar approach in creating discovery learning environment in their mechanics classes. They *showed* counter-examples to their students and considered them as a chance to learn from mistakes. They claim that for counter-examples to be effective they 'must be recognized to be meaningful and acceptable and must be suggestive, to lead a learner to correct understanding' (Horiguchi & Hirashima, 2001). Mason and Watson (2001) used a method of so-called boundary examples, which suggested creating by students examples to *correct* statements, theorems, techniques, and questions that satisfied their conditions. 'When students come to apply a theorem or technique, they often fail to check that the conditions for applying it are satisfied. We conjecture that this is usually because they simply do not think of it, and this is because they are not fluent in using appropriate terms, notations, properties, or do not recognise the role of such conditions' (Mason and Watson, 2001). In our study, often not the lecturer but *the students* were asked to create and show counter-examples to *incorrect* statements, so the students themselves established a conflict in their minds. The students were actively involved in creative discovery learning that stimulated development of their advanced statistical thinking.

The Study

The students from a first-year course 'Probability Theory and Applications' were given mathematical statements and asked to create counter-examples to disprove these statements. They had enough knowledge to do that. However, for most of the students this kind of activity was absolutely new, very challenging and even created psychological discomfort and conflict for a number of reasons. In the beginning some of the students could not see the difference between "proving" that the statement is correct by an example and disproving it by an example. It agrees with the following observation from Selden & Selden (1998): 'Students quite often fail to see a single counter-example as disproving a conjecture. This can happen when a counter-example is perceived as "the only" one that exists, rather than being seen as generic'. To illustrate the idea of disproving by a counter-example it might be helpful to use non-mathematical examples first. For instance, it might be discussed with students: What does it take to disprove the statements like 'all Scandinavians are blond' or 'there are no numbers such that when they are spelled they contain the letter "a". Apart from the activity on using counter-examples the students were also given some paradoxes and were asked to explain them.

In our study we did not use 'pathological' cases. All exercises given to the students were within their knowledge and often were related to their common misconceptions.

Below are examples of the incorrect statements to be disproved by counter-examples and the paradoxes to be explained that were discussed with the students.

Counter-Examples

Use counter-examples to disprove the following incorrect statements.

1) Pairwise independence of events implies their independence.

2) a) If events A and B are independent, then they are conditionally independent.

b) If events A and B are conditionally independent, then they are independent.

3) Uncorrelated random variables are independent.

a) Consider the case of discrete random variables.

b) Consider the case of continuous random variables.

4) Pairwise independence of random variables implies their mutual independence.

Paradoxes

We consider the following problems as paradoxes because the correct answer to each of them contradicts intuition.

Galton's paradox. (Grimmett & Stirzaker, 2004, p. 14).

You flip three fair coins. At least two results are alike (the same). There is 50-50 chance that the third one is a head or a tail. Therefore the probability that all three results are alike equals 0.5. Do you agree?

Simpson's paradox. (Grimmett & Stirzaker, 2004, p. 19).

A doctor has performed clinical trials to determine the relative efficacies of two drugs, with the following results:

Table 1

Results of Drug Treatment

	Women		Men	
	Drug 1	Drug 2	Drug 1	Drug 2
Success	200	10	19	1000
Failure	1800	190	1	1000
Total	2000	200	20	2000

The success rate of Drug 1 is $219/2020 \approx 0.108$ and of Drug 2 is $1010/2200 \approx 0.459$, so the overall success rate is greater for Drug 2.

Among women the success rates are:

$200/2000 = 0.1$ for Drug 1 and $10/200 = 0.05$ for Drug 2.

Among men the success rates are:

$19/20 = 0.95$ for Drug 1 and $1000/2000 = 0.5$ for Drug 2.

So the success rates are greater for Drug 1 when the proportions are calculated for men and women separately.

Which drug is better?

Monty Hall paradox. (Grimmett & Stirzaker, 2004, p. 12).

Suppose you are in a game show, and you are given the choice of three doors of which one contains a prize. The other two contain gag gifts like a goat or a donkey. You pick a door, say, No. 1. The host (who knows what behind the doors) opens door 3, which has a donkey. He then says to you, "Do you want to pick door 2?". Is it to your advantage to switch your choice? That is, will your probability of winning increase if you switch to door 2?

The intuition of many students tells them that switching the door does not change the probability of winning. Actually this probability increases from $\frac{1}{3}$ to $\frac{2}{3}$.

Prisoners' paradox. (Grimmett & Stirzaker, 2004, p. 11). There are three prisoners, A, B, and C. The warden tells them that two of them will be released and one will be executed. But he is not permitted to reveal to any prisoner the fate of that prisoner.

A asks the warden to tell him the name of one of his cohort who will be released. The warden obliges and says, "B will be released." Assume that the warden tells the truth.

a) What are A's and C's respective probabilities of dying now?

b) If A could switch fates with C now, should he?

This paradox is similar to Monty Hall paradox. Contrary to what intuition tells us, the conditional probabilities of dying are different for A and C ($\frac{1}{3}$ and $\frac{2}{3}$ respectively).

St Petersburg's paradox. (Grimmett & Stirzaker, 2004, p. 55).

In a game of chance, a player pays a fixed fee to enter, and then a fair coin is tossed repeatedly until a head appears ending the game. If the first head appears after n tosses, then the player gets $\$2^n$.

a) What is the expected win of a player?

b) What is the "fair" entrance fee?

The answer for both questions is ∞. It seems that any high fee is worth paying to enter this game, which contradicts common sense.

The Questionnaire

After several weeks of using such exercises in class the students were given the following questionnaire to investigate their attitudes towards the usage of counter-examples and paradoxes in learning/teaching.

Question 1.

Do you find counter-examples and paradoxes useful for understanding this course?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 2.

Do you feel confident using counter-examples and paradoxes?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 3.

Do you find this strategy effective?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 4.

Would you like this kind of activity to be a part of assessment?

a) Yes Please give the reasons:

b) No Please give the reasons:

Findings from the Questionnaire

The statistics from the questionnaire are presented in the following table.

Table 2

Summary of Findings from the Questionnaire

Number of Students	Question 1 Useful?		Question 2 Confident?		Question 3 Effective?		Question 4 Assessment?	
	Yes	No	Yes	No	Yes	No	Yes	No
11	10	1	7	4	11	0	4	7
100%	91%	9%	64%	36%	100%	0%	36%	64%

The majority of the students found counter-examples and paradoxes useful for understanding the course. The typical comments from those students were as follows:

- they are both entertaining and informative;
- they are helpful because we can look back at them when we do assignments;
- we go though reasoning of counter-examples and paradoxes that helps understanding the course.

About 2/3 of the students (64%) felt confident using counter-examples and 36% did not. The ones who answered 'no' to the question about confidence provided the following comments:

- counter-examples are difficult;
- sometimes they are confusing;
- I need more practice with them.

All surveyed students found the method of counter-examples effective and provided the following typical comments:

- it improves my understanding of probability and random variables;
- it builds my logical skills;
- it strengthens my thinking ability.

About 2/3 of the students (64%) did not like the idea of counter-examples being a part of assessment. In their comments they wrote that they could cope only with simple counter-examples in assessment or with counter-example problems only in home assignments but not in class tests. To some extent this last result contradicts the responses to questions 2 and 3, where many students indicated that they felt confident using counter-examples and that they considered the method effective.

Conclusions and Recommendations

The statistical results of this study show positive attitudes of the students towards using paradoxes and counter-examples as a pedagogical strategy in a first-year course in probability and random variables. All students surveyed stated that the pedagogical strategy was effective. The majority of the students (91%) stated that the strategy was useful for understanding the course. Many students commented that this method helped them improve their logical skills and critical thinking and made the learning environment more creative and entertaining. Though most of the surveyed students did not encounter counter-examples in past and often found them challenging, they also found them useful and effective and wanted to practise more with such problems.

As with any other case study the question is: to which extent can the results of the study be generalised? The question remains regardless of the number of students surveyed in the study. It doesn't matter whether there are 11 students or 20 students or 50 students in a class – there is only one lecturer and one learning environment and the number of students surveyed is drop in the ocean compared to the number of students in the world studying the first-year university probability course. In addition in this particular class many students were in year 2 and 3 of their studies, therefore they had a better mathematical background than typical year 1 students. It makes the study a bit biased. So the results of the study can be treated as an invitation for colleagues to try the suggested strategy with their own students and see how it works with them. It definitely worked for our students.

As the first step in introducing counter-examples the authors recommend that a lecturer provides a paradox or a counter-example and asks the students to explain or justify it. Next the students can be asked to create their own counter-examples for a given incorrect statement. And finally, the lecturer can ask the students to decide whether a given mathematical statement is correct, so the students have to come up with a proof to show that the statement is true, or with a counter-example to show that the statement is wrong. In a one-semester course we tried to lead the students through these three steps with a certain amount of success. But we observed that many students needed a lot more practice to succeed and feel more confident in this area.

Further Study

We would like to extend the study to measure the effectiveness of this pedagogical strategy on the students' exam performance on the questions that require good understanding of concepts, not just manipulations and techniques. We plan to compare the performance of two groups of students with similar backgrounds. In one group we will extensively use counter-examples and paradoxes, with the other group being the control group. Then we will use statistical methods to establish whether the difference is significant or not.

References

Bermudez, C.G. (2004). Counterexamples in calculus teaching. *Paper presented at the 10th International Congress on Mathematics Education (ICME-10).* Copenhagen, Denmark.

Gelbaum, B.R. & Olmstead, J.M.H. (1964). Counterexamples in Analysis. San Francisco: Holden-Day.

Grimmett, G. & Stirzaker, D. (2004). *Probability and Random Processes.* (3rd ed.). New York: Oxford University Press.

Gruenwald, N. & Klymchuk, S. (2003). Using counter-examples in teaching calculus. *The New Zealand Mathematics Magazine. 40*(2), 33-41.

Horiguchi, T. & Hirashima, T. (2001). The role of counterexamples in discovery learning environment: Awareness of the chance for learning. *Proceedings of the 1st International Workshop on Chance Discovery* (pp. 5-10). Matsue, Japan.

Irwin, K. (1997). What conflicts help students learn about decimals? *Proceedings of the International Conference of Mathematics Education Research Group of Australasia* (pp. 247-254). Rotorua, New Zealand.

Klymchuk, S. (2005). Counter-examples in teaching/learning of calculus: Students' performance. *The New Zealand Mathematics Magazine. 42*(1), 31-38.

Klymchuk, S. (2004). *Counter-examples in calculus.* New Zealand: Maths Press.

Mason, J. & Watson, A. (2001). Getting students to create boundary examples. *MSOR Connections, 1*(1), 9-11.

Peled, I. & Zaslavsky, O. (1997). Counter-examples that (only) prove and counter-examples that (also) explain. *Focus on Learning Problems in Mathematics. 19*(3), 49-61.

Piaget, J. (1985). *The Equilibrium of Cognitive Structures.* Cambridge, MA: Harvard University Press.

Romano, J.P. (1993). *Counterexamples in probability and statistics.* Chapman & Hall/CRC.

Selden, A. & Selden, J. (1998). The role of examples in learning mathematics. *The Mathematical Association of America Online.* Retrieved February 2, 2007, from website: www.maa.org/t_and l/sampler/rs_5.html

Stoyanov, J.M. (1997). *Counterexamples in probability.* (2nd ed.) England: Wiley.

Swan, M. (1993). *Becoming numerate: Developing conceptual structures.* In Willis (Ed), *Being numerate: What counts?* (pp. 44-71). Hawthorne VIC: Australian Council for Educational Research.

Swedosh, P. & Clark, J. (1997). Mathematical misconceptions – can we eliminate them? *Proceedings of the International Conference of Mathematics Education Research Group of Australasia* (pp. 492-499). Rotorua, New Zealand.

Tall, D. (1991). The psychology of advanced mathematical thinking. In Tall (Ed), *Advanced Mathematical Thinking* (pp. 3-21). Kluwer: Dordrecht.

Tall, D., Gray, E., Bin Ali, M., Crowley, L., DeMarois, P., McGowen, M., et al. (2001). Symbols and the bifurcation between procedural and conceptual thinking. *Canadian Journal of Science, Mathematics and Technology Education, 1*, 81–104.

Wise, J.L. & Hall, E.B. (1986). *Counterexamples in probability and real analysis.* New York: Oxford University Press.

Zaslavsky, O. & Ron, G. (1998). Students' understanding of the role of counter-examples. *Proceedings of the 22nd Conference of the International Group for the Psychology of Mathematics Education. 1*, 225-232. Stellenbosch, South Africa.

Zevenbergen, R. & Begg, A. (1999). *Theoretical framework in educational research.* In Coll, R.K., et al (Eds) *SAME Papers* (pp. 170-185). New Zealand.