USING PARADOXES AND COUNTEREXAMPLES IN TEACHING PROBABILITY:

A PARALLEL STUDY.

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The paper presents and analyses the attitudes of first-year university science and engineering students towards using paradoxes and counterexamples as a pedagogical strategy in teaching and learning of probability. The research question was to investigate the effectiveness of this pedagogical strategy from a student's point of view. It is based on a parallel study conducted at two universities – one in New Zealand and the other in Germany. The vast majority of the students reported that they found the method of using paradoxes and counterexamples to be useful and effective.

Keywords: paradoxes; counterexamples; probability; teaching; learning

INTRODUCTION

Research on using paradoxes and counterexamples in teaching and learning of probability at the university level is not a *terra incognita* but it has a limited data base. Earlier studies investigated the usage of paradoxes on students' motivation of learning probability and statistical concepts (Movshovitz-Hadar & Hadass, 1990; Shaughnessy, 1977; Wilensky, 1995). Sowey (2001) successfully used so called 'striking demonstrations' in teaching probability and statistics 'to help make the subject memorable'. They had different forms – 'a unifying formulation of at-first-sight diverse results, as a counterintuitive (but true) proposition, as a logical paradox, as a counterexample to a seemingly general principle, as an analogy with something already familiar to the student, as a vivid geometric, numeric, or graphical illustration of some abstract principle or algebraic theorem, or as an unusual and attention-gripping application of familiar statistical tools'. Leviatan (2002) developed a three-tier teaching method introducing so-called judgment problems, probabilistic puzzles and classical paradoxes in a new topic. She argued that 'properly introduced, paradoxes can play a very useful role in the classroom as they serve as leverage to fruitful discussions, and provoke deeper thinking about the (not always intuitive) probabilistic ideas… In addition to their role in clarifying the new concepts, giving better insight to the theory, removing all sorts of potential conflicts between intuition and theory, and helping form new and correct intuitions, they introduce a new dimension to the course. Students cannot remain passive; they are forced to be sharp and witty, to constantly test themselves and to learn to expect the unexpected' (Leviatan, 2002). But she also warned that this powerful pedagogical strategy should be used carefully – 'introducing paradoxes in class carries potential danger: it may result in a feeling of insecurity when the conflict between the mathematical solution and the intuition (or between two seemingly correct mathematical solutions) seems unresolvable' (Leviatan, 2002). Lesser (2006) made a connection between so called counterintuitive results in probability and statistics and statistical literacy: 'Some fundamental and/or famous results in probability and statistics are counterintuitive…Many counterintuitive results can be used effectively to capture students' attention and force them to engage with concepts in a more sustained way and with a deeper understanding…Besides losing the inherent benefits that come from challenges to sharpen one's conceptual knowledge, teachers (or textbook writers) who might be tempted to avoid all counterintuitive examples would be doing their students a further disservice, for several counterintuitive phenomena actually occur in the real world, and awareness of them is therefore needed for statistical literacy. For example, Simpson's Paradox is listed as essential for citizenship by the National Council on Education and the Disciplines (2001)' (Lesser, 2006). Batanero, Contreras, Fernandes and Ojeda (2010) successfully applied paradoxical games as a didactic tool to train teachers in probability. They utilized some 'classical paradoxes (Székely, 1986) to build didactic situations that serve to provoke the teachers' didactic reflection' (Batanero, Contreras, Fernandes, & Ojeda, 2010). They referred to the following components of teachers' professional knowledge: epistemology, cognition, teaching resources and techniques, affect and interaction. By affect they understood the 'ability to engage students' interest and take into account the students' attitudes and beliefs' (Batanero et al., 2010). Schield (1999) investigated the pros and cons of teaching the Simpson's paradox in a university introductory statistics course. He argued that the Simpson's paradox is 'vitally important' for a number of reasons although 'it is not easy to understand the reasons for - much less the cause of - a reversal of an association, i.e., Simpson's Paradox' (Schield, 1999, p.110). He concluded that 'if students are to understand proper inductive reasoning about causality in observational studies, they must understand Simpson's Paradox' (Schield, 1999, p.110).

Counterexamples are a powerful and effective tool for scientists, researchers and practitioners. According to Stoyanov (1986) counterexamples 'define the power, the wideness, the depth, the degree of nontriviality, and of course, the beauty of the theory' (Stoyanov, 1986, p. 281). Several books are written on counterexamples in probability and statistics (Romano & Siegel, 1986; Stoyanov, 1997; Wise & Hall, 1993). A number of studies about the effect of available counterexamples on conditional inference making were reviewed in (Politzer & Bourmaud, 2002). The role of counterexamples in causal conditional reasoning was investigated in a more recent study by Verschueren, Schaeken and Ydewalle (2005). The authors described and tested two accounts of everyday conditional reasoning - a probabilistic account and a mental model account. They claim that 'the analytic core of the reasoning mechanism lies not so much in the retrieval of counterexamples but in the way counterexamples are taken into account' (Verschueren, Schaeken & Ydewalle, 2005, p. 248). In particular, 'in the mental models process, inference acceptance depends on the retrieval of counterexamples. When a counterexample is taken into account, responders reject the default conclusion. When the counterexample is not taken into account, responders accept the default conclusion' (Verschueren, Schaeken & Ydewalle, 2005, p. 248). Yang and Chang in their study on strategies of making counterexamples by pre-service and in-service teachers (Yang & Chang, 2010) found that the teachers need guidance in creating examples. 'While in-service teachers were asked to make counterexamples, few of them could make an example to show the inconsistent relations between mean and probability of two distributions in about ten minutes. For encouraging pre-service teachers to make examples, the first author guided them to discuss the conditions which should be satisfied by examples. After about fifteen minutes, many of them could make an example' (Yang & Chang, 2010). They also noted that many teachers tend to treat counterexamples as pathological or special cases. 'After this teacher presented and explained his counterexample, there were still many teachers who refused to accept that using the mean was not appropriate. They thought those counterexamples were just only special cases, they couldn't represent all situations' (Yang & Chang, 2010). It is consistent with findings from some studies on using counterexamples in teaching and learning of calculus. Selden and Selden (1998) have articulated these ideas: 'Students quite often fail to see a single counterexample as disproving a conjecture. This can happen when a counterexample is perceived as 'the only one that exists', rather than being seen as generic.' A similar observation was reported by Zaslavsky and Ron (1998): 'Students often feel that a counterexample is an exception that does not really refute the statement in question' (p. 4-231).

In this study, we attempted to analyse the effectiveness, from a student's point of view, of the usage of paradoxes and counterexamples as a pedagogical strategy in the teaching and learning of probability in a first-year university course.

THEORETICAL FRAMEWORK

The theoretical framework in this study was based on Piaget's notion of cognitive conflict (Piaget, 1985) and the notion of 'pivotal-bridging example' introduced by Zazkis and Chernoff (2008). The goals of the study were to investigate students' attitudes towards the usage of paradoxes and counterexamples in teaching and learning of probability and analyse the effect of cognitive conflict and how students resolve it from the point of view of the notion of the pivotal-bridging example. A cognitive conflict arises when a student encounters information containing some sort of contradiction or inconsistency with their own ideas or experience. However, as Zazkis and Chernoff (2008) pointed out ‘inconsistency of ideas presents a *potential conflict*, it becomes a *cognitive conflict* only when explicitly invoked, usually in an instructional situation’ (p. 196). In our study, the students were asked to explain paradoxes and create counterexamples to incorrect statements, that is the students themselves established a conflict in their mind. However, as Zazkis and Chernoff (2008) pointed out ‘researchers in mathematics education are well aware that learners may possess contradicting ideas without facing or acknowledging a conflict. As such, a counterexample, when presented to a learner, may not create a cognitive conflict; it may be simply dismissed or treated as exception’ (Zazkis & Chernoff, 2008). To analyse how the students resolved their conflict (exposed by a lecturer or created by themselves) we used the notion of ‘pivotal-bridging example’ (Zazkis & Chernoff, 2008). ‘An example is pivotal for a learner if it creates a turning point in the learner’s cognitive perception or in his or her problem solving approaches; such examples may introduce a conflict or may resolve it.’ (Zazkis & Chernoff, 2008, p. 197).

THE STUDY

PARTICIPANTS. Two groups of students were exposed to the practice of using paradoxes and counterexamples as a pedagogical strategy in teaching and learning of probability. The first group comprised 11 students from the Auckland University of

Technology, New Zealand studying a first-year course 'Probability Theory and Applications'. The students from the first group were majoring in science and applied mathematics. The second group comprised 30 students from Wismar University of Applied Sciences, Technology, Business and Design, Germany studying probability theory as part of their first-year mathematics course. Those students were majoring in engineering. At the end of the semester the students from both groups were given a questionnaire to investigate their attitudes towards the usage of paradoxes and counterexamples in teaching and learning of probability. All 11 students from the first group and 23 students out of 30 from the second group answered the questionnaire. The students in the German university received the questionnaire in German. The students' responses in both groups were very similar, so we combined them. The response rate therefore was 34/41, which is 83%. The participation in the study was voluntary, so we had a self-selected sample.

USING PARADOXES AND COUNTEREXAMPLES IN TEACHING. In our study we did not use 'pathological' cases. All paradoxes and incorrect statements for creating counterexamples that were given to the students were within their knowledge and often related to their common misconceptions. Paradoxes and incorrect statements were given to the students for a group discussion after solving a number of routine exercises on a topic. The intention was to enhance the students' understanding of probability concepts, increase their motivation to study the subject and engage their emotions as both paradoxes and counterexamples were non-routine exercises with counterintuitive and surprising answers. The students' knowledge of probability concepts was assessed by written home assignments, a mid-semester test and a final examination. Paradoxes were not included in the assessment. Exercises on creating counterexamples to incorrect statements were included in home assignments that were part of the assessment.

One of the paradoxes discussed with the students was Simpson's paradox.

Simpson's paradox. (Grimmett & Stirzaker, 2004, p. 19).

A doctor has performed clinical trials to determine the relative efficacies of two drugs, with the results presented in Table 1 below:

Table 1. Results of drug treatment.

	Women		Men	
	Drug 1	Drug 2	Drug 1	Drug 2
Success	200	10	19	1000
Failure	1800	190	1	1000
Total	2000	200	20	2000

a) Calculate the overall success rate of each drug.

b) Calculate the success rate of each drug for men.

c) Calculate the success rate of each drug for women.

d) Which drug is better?

The students had no difficulties in answering questions a) – c). They had a group discussion, which showed that they were puzzled by the results of their calculations. Clearly their intuition said that the same drug should work better in all cases. Some of the students said: “perhaps my calculations were wrong”. They could not reach a common conclusion for question d). The lecturer commented on the importance of causal relationships, which helped the students to realise that Drug 1 was better (since it had higher success rates for men and women separately). Then the students discussed other cases when the Simpson’s paradox might arise. We observed an increased interest and motivation to the subject in both New Zealand and German groups of students when we mentioned the fact that the Simpson’s paradox is considered so important for statistical literacy in the USA that it is listed as essential for citizenship by the National Council on Education and the Disciplines (2001).

The other paradoxes discussed in class include the Galton, Monty-Hall, St Petersburg and Prisoner’s paradoxes. Their descriptions can be found in (Grimmett & Stirzaker, 2004). The lecturer commented on the paradoxes and provided some hints that helped the students to resolve them.

Apart from paradoxes we also used counterexamples in teaching the courses. Below are examples of incorrect statements that the students had to disprove by creating counterexamples.

Statement 1. Pairwise independence of events implies their mutual independence.

Statement 2. If events A and B are independent, then they are conditionally independent.

Statement 3. If events A and B are conditionally independent, then they are independent.

Statement 4. Uncorrelated random variables are independent.

Statement 5. Pairwise independence of random variables implies their mutual independence.

The notion of a counterexample was discussed with the students as many of them had never heard about it before. In the beginning the students had difficulties in creating counterexamples, so some guidance was needed from the lecturer. For example, for Statement 4 the lecturer introduced the joint distribution of random variables X and Y presented in Table 2 below:

Table 2. Lecturer's counterexample to Statement 4.

X* \ *Y	0	1	2

0	$\frac{1}{4}$	0	$\frac{1}{4}$
1	0	$\frac{1}{2}$	0

Then the students verified that X and Y make a counterexample to Statement 4. They had a group discussion of the result and were asked to make their own counterexamples. A successful student's attempt is presented in Table 3 below:

Table 3. Student's counterexample to Statement 4.

X* \ *Y	−1	0	1
−1	$\frac{1}{3}$	0	$\frac{1}{3}$
1	0	$\frac{1}{3}$	0

THE QUESTIONAIRE. The students' attitudes towards the usage of paradoxes and counterexamples in teaching and learning of probability were investigated using the following questionnaire.

Question 1.

Do you find paradoxes and counterexamples useful for understanding this course?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 2.

Do you feel confident using paradoxes and counterexamples?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 3.

Do you find this strategy effective?

a) Yes Please give the reasons:

b) No Please give the reasons:

Question 4.

Would you like this kind of activity to be a part of assessment?

a) Yes Please give the reasons:

b) No Please give the reasons:

FINDINGS FROM THE QUESTIONAIRE. The statistics from the questionnaire are presented in Table 4 below:

Table 4. Summary of findings from the questionnaire.

Number of Students	Question 1 Useful?		Question 2 Confident?		Question 3 Effective?		Question 4 Assessment?	
	Yes	No	Yes	No	Yes	No	Yes	No
34	31	3	16	18	30	4	11	23
Percentage	91%	9%	47%	53%	88%	12%	32%	68%

Questions 1 and 3 were the main questions of the questionnaire. Obviously we could not use such notions as 'cognitive conflict' or 'pivotal-bridging example' in the questionnaire. Instead we explained to the students what we meant by "effective method": it makes you think about some aspects of probability that you never considered before; it opens your eyes on the importance of conditions in rules and theorems; it reveals your misconceptions; it forces you to pay attention to every detail; it enhances your conceptual understanding of probability. The vast majority of the students reported that they found the method of using paradoxes and counterexamples to be useful (91%) and effective (88%). The common comments from the students who answered 'Yes' to question 1 on the usefulness and question 3 on the effectiveness were as follows:

> 'They are both entertaining and informative; they are helpful because we can look back at them when we do assignments; we go through reasoning of counterexamples and paradoxes that helps understanding the course; helps to

> overcome superficial knowledge; good illustration of theory; it makes the use of probabilities more clear; examples explain complex connections; it builds my logical skills; motivates to be critical and to overcome stupid solution schemes; trains to reconsider – paradoxes are more difficult than counterexamples; it supports deeper thinking and understanding; examples are illustrative and interesting and good for logical thinking; they support the understanding of problems; it strengthens my thinking ability; it is interesting to consider different solutions; better understanding; it improves my understanding of probability and random variables; helps understanding and logical thinking; better feeling (sense) for the topic; paradoxes at first look confusing, but then revealing; different aspects are shown and broaden knowledge; promotes the understanding which is useful for the future; this teaching principle was very helpful for me; the illustration helps in understanding the whole; supports interest and deep thinking; encourage to think'.

We used a combination of a convenience and a judgement sampling methods to select 9 students for subsequent interviews. From the interviews we noticed that explaining a paradox and showing a counterexample to an incorrect statement by a lecturer did not always create a cognitive conflict for a student. In the cases when it did create a conflict it sometimes remained unresolved. Some guidance on 'how' and 'why' was helpful and effective, from a student's point of view, for conflict resolution. A typical example of such guidance was asking the students to create at least one more counterexample to the given incorrect statement to demonstrate that the discussed counterexample was not just an exception.

There were several discussions with all students about the role of intuition in solving probability problems. The students realised that they could not rely only on intuition without applying appropriate mathematical techniques, since this might be misleading and create misconceptions. We taught our students to analyse each probability statement critically by trying to prove it or disprove it with a counterexample. Reflecting on the students' behaviour in class discussions, their comments in the questionnaire and interviews and applying the notion of pivotal-bridging example (Zazkis & Chernoff, 2008) we can state that for the majority of the students the paradoxes and counterexamples shown by the lecturer or constructed by themselves were pivotal examples in the sense that they created 'a turning point in the learner's cognitive perception or in his or her problem solving approaches' (Zazkis & Chernoff, 2008, p. 197). When paradoxes and counterexamples helped the students eliminate their misconceptions and resolve their conflict they became bridging examples because they served 'as a bridge from learner's initial (naïve, incorrect or incomplete) conceptions towards appropriate mathematical conceptions' (Zazkis & Chernoff, 2008, p. 197).

About half (47%) of the students eventually felt confident using paradoxes and counterexamples and the others (53%) did not. More than 2/3 of the students (68%) did not like the idea of using paradoxes and counterexamples in assessment. The main reasons indicated by the students were the time factor ('counterexamples and paradoxes are difficult to resolve under time pressure', 'too complex and time consuming for examinations') and the stress in the exam situation ('for most students normal maths is stressful enough, weak students would be desperate', 'under examination stress we behave rationally, and paradoxes are against rationality'). To some extent this contradicts the responses to questions 1 and 3, where the vast majority of the students indicated that they found paradoxes and counterexamples useful and effective. Clearly the students were more concerned about their exam performance rather than acquiring useful skills.

CONCLUSIONS

The statistics results and numerous students' comments from the study showed that the students were very positive about the usage of paradoxes and counterexamples in the teaching and learning of probability. The vast majority of them reported that they found this pedagogical strategy to be useful (91%) and effective (88%) in the sense that from their point of view it advanced one or more of the following: thinking about some aspects of probability that they never considered before; understanding the importance of conditions in rules and theorems; revealing their misconceptions; paying attention to every detail; enhancing their conceptual understanding of probability. Thirty six percent of the students made the comment that this pedagogical strategy helped them to improve their logical thinking. As the first step in this teaching approach we would recommend that a lecturer provides a paradox or a counterexample and asks students to explain or justify it. Next the students could be asked to create their own counterexamples to a given incorrect statement. And finally, the lecturer can ask the students to decide whether a given mathematical statement is correct, so the students have to come up with a proof that the statement is true, or with a counterexample to show that the statement is false. There are many ways of using counterexamples as a pedagogical strategy:

- giving students a mixture of correct and incorrect statements;
- asking students to create their own wrong statements and counterexamples to them;
- making a deliberate mistake in a lecture;
- asking students to spot an error on a certain page of their textbook;
- giving students bonus marks towards their final grade for creating excellent counterexamples;
- including into assessment the questions that require construction of counterexamples.

In a one-semester course we tried to lead the students through some of these activities with a certain amount of success. Taking into account that the vast majority of the students' gave very positive feedback on the usefulness and effectiveness of the usage of paradoxes and counterexamples in teaching and learning of probability it is clear that not only top students but the majority of average and weak students were enthusiastic about studying paradoxes and counterexamples in the probability course. It gives us confidence to recommend this pedagogical strategy to our colleagues who teach probability in a first-year university course.

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