

THREE OPEN PROBLEMS ON THE WIJSMAN TOPOLOGY*

JILING CAO

ABSTRACT. Since it first emerged in Wijsman's seminal work [29], the Wijsman topology has been intensively studied in the past 50 years. In particular, topological properties of Wijsman hyperspaces, relationships between the Wijsman topology and other hyperspace topologies, and applications of the Wijsman topology in analysis have been explored. However, there are still several fundamental open problems on this topology. In this article, the author gives a brief survey on these problems and some up-to-date partial solutions.

1. INTRODUCTION

Let (X, d) be a metric space, and 2^X be the collection of all non-empty closed subsets of X . There are many ways to equip a topology \mathcal{T} on 2^X such that $(X, \mathcal{T}(d))$ is embedded into $(2^X, \mathcal{T})$ as a closed subspace via the mapping $x \mapsto \{x\}$, where $\mathcal{T}(d)$ is the topology on X induced by the metric d . A unified approach to topologize 2^X discussed in the monograph [3] is using distance functionals $d(\cdot, A) : X \rightarrow \mathbb{R}^+$ for sets $A \in 2^X$, where $d(\cdot, A)$ is defined such that for each $x \in X$,

$$d(x, A) = \inf\{d(x, a); a \in A\}.$$

The weakest topology on 2^X such that all functionals $d(\cdot, A)$, $A \in 2^X$, are continuous is called the *Wijsman topology* and denoted by $\mathcal{T}_{W(d)}$. Correspondingly, we call $(2^X, \mathcal{T}_{W(d)})$ the *Wijsman hyperspace* of (X, d) . The Wijsman topology is formally introduced in Lechicki and Levi [25], but it can be tracked back to the seminal work of Wijsman [29], where R. A. Wijsman considered a mode of convergence for sequences of closed sets when he studied some optimum properties of sequential probability ratio test in 1960's.

The Wijsman topology is closely related to the other two well studied hyperspace topologies on 2^X : the *Hausdorff metric topology* $\mathcal{T}_{H(d)}$ and the *Viatoris topology* \mathcal{T}_V . Firstly, it is easily verified that the (extended) Hausdorff distance $H_d(A, B)$ between two members $A, B \in 2^X$ can be defined as

$$H_d(A, B) = \sup\{|d(x, A) - d(x, B)| : x \in X\}.$$

Thus, the Hausdorff metric topology is just the topology of uniform convergence on 2^X under the identification $A \leftrightarrow d(\cdot, A)$, while the Wijsman topology is the

*This article was based on the author's talk at the 16th Galway Topology Colloquium, Galway, Ireland, 8–10 July 2013. The author wishes to thank the School of Mathematics, Statistics and Applied Mathematics at NUI Galway for its hospitality during his visit in July 2013.

2010 *Mathematics Subject Classification*. Primary 54B20; Secondary 54D15, 54E50, 54E52.

Keywords. Baire, completely metrizable, completeness-type properties, non-separable, normal, Wijsman hyperspace, Wijsman topology.

topology of pointwise convergence on 2^X under the same identification. Secondly, it is known that the Vietoris topology on 2^X is given by

$$\mathcal{T}_V = \sup \{ \mathcal{T}_{W(\varrho)} : \varrho \text{ is a metric on } X \text{ equivalent to } d \}.$$

The reader can find more details on relationships among these and other hyperspace topologies in [3]. However, it turns out that the Wijsman topology is more intractable than both of the Hasudorff metric and Vietoris topologies.

In the past 50 years, the Wijsman topology and convergence under this property has been intensively studied. For example, possible extensions of Wijsman's results in infinite-dimensional Banach spaces have been considered, refer to [4]. In addition, various properties of the Wijsman topology, conditions for the coincidence of the Wijsman topology with other hyperspace topologies and function spaces equipped with the Wijsman topology have also been investigated, refer to [3], [4] and [16]. The Wijsman topology in the context of quasi-metric spaces has been studied in [10] and [27]. In the study of properties of the Wijsman topology, several techniques including game-theoretic approaches, special embeddings and construction of special power spaces etc., have been employed. Despite of these efforts, several fundamental problems still remain unsolved.

In this short article, the author will discuss three basic open problems on the Wijsman topology and some open question associated with these problems, studied by him and his co-authors in the past 10 years. The first problem is when the Wijsman topology induced by a metric is normal, the second problem is whether the Wijsman topology induced by a completely metrizable metric has some completeness-type properties, and the third one is to find characterization as when the Wijsman topology has the Baire property in terms of some properties of the underlying metric space. For each of these problems, the author will give a brief survey on its background and then also some up-to-date partial solutions. The reader can find more details of these problems and some of their associated questions from the listed references, particularly from [5], [7], [8], [9] and [11], respectively. Note that the author has no intention to give a comprehensive and up-to-date survey on the Wijsman topology. For undefined notation and background knowledge on the Wijsman topology, refer to [3], [4] and [16].

2. THE NORMALITY PROBLEM AND ASSOCIATED QUESTIONS

First of all, note that all Wijsman topologies are weak topologies, and thus they are Tychonoff. However, not all Wijsman topologies are normal. To see this, let d be the 0-1 metric on a nonempty set X with $|X| = \aleph_1$. As observed in Remark 3.1 of [13], $(2^X, \mathcal{T}_{W(d)})$ is homeomorphic to $\{0, 1\}^{\aleph_1} \setminus \{\mathbf{0}\}$, where $\{0, 1\}$ is equipped with the discrete topology and $\mathbf{0}$ is the constant function with value 0. It is known that $\{0, 1\}^{\aleph_1} \setminus \{\mathbf{0}\}$ is not normal, and consequently, $(2^X, \mathcal{T}_{W(d)})$ is not a normal space. Thus, the following natural problem arises.

Problem 2.1 ([8]). *Let (X, d) be a metric space. When is the Wijsman hyperspace $(2^X, \mathcal{T}_{W(d)})$ a normal space?*

A classical result of Lechicki and Levi in [25] states that for a metric space (X, d) , $(2^X, \mathcal{T}_{W(d)})$ is metrizable if and only if (X, d) is separable. As an immediate consequence, if (X, d) is separable, then $(2^X, \mathcal{T}_{W(d)})$ is normal. However, it is

unclear if the converse holds. Indeed, the following problem was posed by Di Maio and Meccariello in 1998.

Problem 2.2 ([16]). *It is known that if (X, d) is a separable metric space, then $(2^X, \mathcal{T}_{W(d)})$ is metrizable and so paracompact and normal. Is the opposite true? Is $(2^X, \mathcal{T}_{W(d)})$ normal if, and only if, $(2^X, \mathcal{T}_{W(d)})$ is metrizable?*

Regarding Problems 2.1 and 2.2, it was conjectured that for a metric space (X, d) , if $(2^X, \mathcal{T}_{W(d)})$ is normal, then (X, d) is separable. In other words, $(2^X, \mathcal{T}_{W(d)})$ is non-normal for a non-separable metric (X, d) . If this conjecture is true, then the answer to Problem 2.2 is affirmative, and also a characterization of normality of $(2^X, \mathcal{T}_{W(d)})$ is derived and thus a solution to Problem 2.1 is obtained. Below, I shall summarize some progress in this direction.

For a metric space (X, d) , define $\text{nlc}(X)$ by

$$\text{nlc}(X) = \{x \in X : x \text{ has no compact neighbourhood in } X\}.$$

The following embedding theorem was proved by Chaber and Pol in 2002.

Theorem 2.3 ([13]). *Let X be a metrizable space such that $w(\text{ncl}(X)) = 2^{\aleph_0}$. Then for any compatible metric d , $\mathbb{N}^{2^{\aleph_0}}$ embeds as a closed subspace in $(2^X, \mathcal{T}_{W(d)})$. In particular, $(2^X, \mathcal{T}_{W(d)})$ contains a closed copy of \mathbb{Q} .*

Following the proof of Theorem 2.3, if $\text{nlc}(X)$ is a non-separable subspace of (X, d) , $(2^X, \mathcal{T}_{W(d)})$ contains a closed copy of \mathbb{N}^{\aleph_1} . This implies that $(2^X, \mathcal{T}_{W(d)})$ is non-normal if $\text{nlc}(X)$ is a non-separable subspace of (X, d) , since \mathbb{N}^{\aleph_1} is non-normal. Particularly, if $(X, \|\cdot\|)$ is a non-separable normed linear space and d is the metric on X induced by $\|\cdot\|$, then $(2^X, \mathcal{T}_{W(d)})$ is non-normal. As a consequence, we derive a partial answer to Problems 2.1 and 2.2, due to Holá and Novotný in 2013.

Theorem 2.4 ([22]). *Let $(X, \|\cdot\|)$ be a normed linear space, and let d be the metric on X induced by $\|\cdot\|$. Then $(2^X, \mathcal{T}_{W(d)})$ is normal if and only if it is metrizable.*

Suppose (X, d) is a non-separable metric space. Then (X, d) contains an ε -discrete subspace Y of size \aleph_1 for some $\varepsilon > 0$. It follows that 2^Y is closed subspace of $(2^X, \mathcal{T}_{W(d)})$. Thus, if one can show that 2^Y is a non-normal subspace of $(2^X, \mathcal{T}_{W(d)})$, then Problems 2.1 and 2.2 would be solved. Indeed, to the author's knowledge, the following question is still unsolved.

Question 2.5. *Let (X, d) be a uniformly discrete and non-separable metric space. Must $(2^X, \mathcal{T}_{W(d)})$ be non-normal?*

Inspired by the work of Keesling in [24], Cao and Junnila [8] explored the possibility whether the non-normal space $\omega_1 \times (\omega_1 + 1)$ is embeddable into the Wijsman hyperspace of a non-separable metric space (X, d) , and they obtained the following result.

Proposition 2.6 ([8]). *Let (X, d) be a non-separable metric space. Then the subspace $2^X \setminus \{X\}$ of $(2^X, \mathcal{T}_{W(d)})$ contains a closed copy of the space $\omega_1 \times (\omega_1 + 1)$.*

Applying Proposition 2.6 and the classical result of Lechicki and Levi in [25] on metrizability of Wijsman hyperspaces, Cao and Junnila were able to derive the following partial answer to Problems 2.1 and 2.2.

Theorem 2.7 ([8]). *Let (X, d) be a metric space. The following are equivalent.*

- (1) $(2^X, \mathcal{T}_{W(d)})$ is metrizable.
- (2) $(2^X, \mathcal{T}_{W(d)})$ is hereditarily normal.
- (3) $2^X \setminus \{X\}$ is a normal subspace of $(2^X, \mathcal{T}_{W(d)})$.

Although the techniques of embeddings shed some light on Problems 2.1 and 2.2, whether the conclusion of Proposition 2.6 can be improved to show that if (X, d) is non-separable, then $(2^X, \mathcal{T}_{W(d)})$ is non-normal is still unclear. This leads to the following question.

Question 2.8. *Let (X, d) be a non-separable metric space. Must $(2^X, \mathcal{T}_{W(d)})$ contain a closed copy of $\omega_1 \times (\omega_1 + 1)$?*

In a recent paper [21], Hernández-Gutiérrez and Szeptycki also considered Problems 2.1 and 2.2. They proved that if (X, d) is a locally separable metric space whose weight is a regular uncountable cardinal, then $(2^X, \mathcal{T}_{W(d)})$ is non-normal. This result also answers partially to Question 2.5. Note that Question 2.8 also suggests the following relevant question:

Question 2.9 ([8]). *Let (X, d) be a metric space. If $(2^X, \mathcal{T}_{W(d)})$ is non-normal, does it contain a closed copy of $\omega_1 \times (\omega_1 + 1)$?*

In [20], Holá gave a partial answer to Question 2.9. In fact, she proved that the answer to Question 2.9 is affirmative when every closed proper ball in X is totally bounded. Consequently, the answer to Problems 2.1 and 2.2 is also affirmative under this assumption.

3. A PROBLEM ON COMPLETENESS-TYPE PROPERTIES AND ASSOCIATED QUESTIONS

The study of completeness-type properties of Wijsman hyperspaces can be tracked back to Effros [17], whose main result can be interpreted as that a Polish space admits a metric whose Wijsman topology is Polish. Beer [2] showed that the Wijsman hyperspace of any separable complete metric space is Polish, and asked whether the Wijsman topology corresponding to an arbitrary compatible metric for a Polish space is necessary Polish. Costantini [14] answered affirmatively this problem, and a simpler proof of Costantini's theorem was given by Zsilinszky [31] in terms of Choquet games. To the author's knowledge, the following problem was due to Beer in an oral communication.

Problem 3.1. *Is complete metrizability of (X, d) (without separability) equivalent to any completeness-type property of $(2^X, \mathcal{T}_{W(d)})$?*

Note that if (X, d) is a non-separable metric space, the Wijsman topology $\mathcal{T}_{W(d)}$ is Tychonoff but non-metrizable. Costantini [15] showed that Čech-completeness is not the right choice to answer Problem 3.1. Indeed, Costantini constructed a 3-valued metric space on the set of real numbers whose Wijsman hyperspace is not Čech-complete. More generally, in a recent paper [9], Cao et al. established the following embedding result.

Theorem 3.2 ([9]). *Every Tychonoff space can be embedded as a closed subspace in the Wijsman hyperspace of a complete metric space (X, d) which is locally \mathbb{R} .*

In the light of Theorem 3.2, in addition to Čech-completeness, any completeness-type property which is closed hereditary, is not the right choice to answer Problem

3.1. This means that, to answer this problem, one has to turn attentions to those completeness-type properties that are not closed-hereditary. Recall that a topological space X is said to be (resp. *countably*) *base compact* with respect to an open base \mathfrak{B} if X is regular such that $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$ for each (resp. countable) centered family $\mathcal{F} \subseteq \mathfrak{B}$, and X is said to be (resp. *countably*) *subcompact* with respect to an open base \mathfrak{B} if X is regular such that $\bigcap_{F \in \mathcal{F}} \overline{F} \neq \emptyset$ for each (resp. countable) regular filterbase $\mathcal{F} \subseteq \mathfrak{B}$. If “regular” is replaced by “quasi-regular” and “base” is replaced by “ π -base”, the resulting spaces are called *almost (countably) base compact*, *almost (countably) subcompact*, respectively. In literature, these properties are called *Amsterdam* properties. For details, refer to [1].

Cao and Junnila [7] tackled Problem 3.1 by considering the Amsterdam properties of Wijsman hyperspaces. They discovered some interesting but peculiar results, controversial to the results for other types of hyperspaces in [5], [6] and [11].

Example 3.3 ([7]). There exists a metric space (X, d) of the first category such that $(2^X, \mathcal{T}_{W(d)})$ is countably base compact with respect to an open base \mathfrak{B} .

Example 3.4 ([7]). There exists a separable metric space (X, d) of the first category such that $(2^X, \mathcal{T}_{W(d)})$ is almost countably subcompact with respect to an open π -base \mathfrak{P} .

Next, we turn our attentions to some completeness-type properties defined by topological games. Let X be a topological space, and \mathfrak{P} be a fixed open π -base. The *Banach-Mazur game* $BM(X)$ is played as follows: Players β and α alternate in choosing elements of \mathfrak{P} , with β choosing first, so that

$$B_0 \supseteq A_0 \supseteq B_1 \supseteq A_1 \supseteq \cdots \supseteq B_n \supseteq A_n \supseteq \cdots$$

Then $B_0, A_0, \dots, B_n, A_n, \dots$ is a *play* in $BM(X)$, and α wins this play if $\bigcap_{n \in \mathbb{N}} A_n (= \bigcap_{n \in \mathbb{N}} B_n) \neq \emptyset$, otherwise, β wins. A *strategy* in $BM(X)$ is a function $\sigma : \mathfrak{P}^{<\omega} \rightarrow \mathfrak{P}$ such that $\sigma(W_0, \dots, W_n) \subseteq W_n$ for all $n \in \mathbb{N}$, and $(W_0, \dots, W_n) \in \mathfrak{P}^{n+1}$. A *tactic* in $BM(X)$ is a function $t : \mathfrak{P} \rightarrow \mathfrak{P}$ such that $t(W) \subseteq W$ for all $W \in \mathfrak{P}$. A *winning strategy* (resp. *tactic*) for α is a strategy (resp. tactic) σ such that α wins every play of $BM(X)$ compatible with σ , i.e., such that $\sigma(B_0, \dots, B_n) = A_n$ (resp. $\sigma(B_n) = A_n$) for all $n \in \mathbb{N}$. A winning strategy (resp. tactic) for β is defined analogously. The space X is called (resp. *weakly*) α -*favorable* [28], if α has a winning tactic (resp. strategy) in $BM(X)$. The space X is called β -*favorable*, if β has a winning strategy in $BM(X)$. Let \mathfrak{B} be an open base for X , and denote

$$\mathcal{E} = \{(x, U) \in X \times \mathfrak{B} : x \in U\}.$$

The *strong Choquet game* $Ch(X)$ is played similarly to the Banach-Mazur game. More precisely, players β and α alternate in choosing $(x_n, B_n) \in \mathcal{E}$ and $A_n \in \mathfrak{B}$, respectively, with β choosing first so that for each $n \in \mathbb{N}$, $x_n \in A_n \subseteq B_n$, and $B_{n+1} \subseteq A_n$. The play $(x_0, B_0), A_0, \dots, (x_n, B_n), A_n, \dots$ is won by α , if $\bigcap_{n \in \mathbb{N}} A_n (= \bigcap_{n \in \mathbb{N}} B_n) \neq \emptyset$; otherwise, β wins. A *strategy* in $Ch(X)$ for α is a function $\sigma : \mathcal{E}^{<\omega} \rightarrow \mathfrak{B}$ such that $x_n \in \sigma((x_0, B_0), \dots, (x_n, B_n)) \subseteq B_n$ for all $((x_0, B_0), \dots, (x_n, B_n)) \in \mathcal{E}^{<\omega}$. A *tactic* in $Ch(X)$ for α is a function $t : \mathcal{E} \rightarrow \mathfrak{B}$ such that $x \in t(x, B) \subseteq B$ for all $(x, B) \in \mathcal{E}$. Winning strategies and tactics in $Ch(X)$ are defined similarly to the ones for the Banach-Mazur game. The space X is *strongly* α -*favorable* [28] (resp. *strongly Choquet* [23]), provided that α has a winning tactic (resp. strategy) in $Ch(X)$.

Regarding the completeness-type properties defined in the above, Piątkiewicz and Zsilinszky [26] established the following results.

Theorem 3.5 ([26]). *Let X be a locally separable metrizable space. The following are equivalent.*

- (1) $(2^X, \mathcal{T}_{W(d)})$ is strongly α -favorable for every compatible metric d on X .
- (2) $(2^X, \mathcal{T}_{W(d)})$ is strongly Choquet for every compatible metric d on X .
- (3) X is completely metrizable.

Theorem 3.6 ([26]). *If X is (weakly) α -favorable and metrizable, then $(2^X, \mathcal{T}_{W(d)})$ is α -favorable for every compatible metric d on X .*

Example 3.7 ([26]). There is a separable metric space (X, d) of the first category such that $(2^X, \mathcal{T}_{W(d)})$ is α -favorable.

Example 3.8 ([26]). There is a non-separable metric space (X, d) of the first category such that $(2^X, \mathcal{T}_{W(d)})$ is strongly α -favorable.

Note that Theorem 3.5 provides an answer to Problem 3.1 in the realm of locally separable metrizable spaces. In the light of Theorems 3.5 and 3.6 as well as Examples 3.7 and 3.8, the following two open questions are interesting.

Question 3.9. *Let X be a completely metrizable space. Must $(2^X, \mathcal{T}_{W(d)})$ be strongly Choquet or strongly α -favorable for every compatible metric d on X ?*

Question 3.10. *Let X be a metrizable space. If $(2^X, \mathcal{T}_{W(d)})$ is strongly Choquet or strongly α -favorable for every compatible metric d on X , must X be completely metrizable?*

Recall that a topological space X is said to be *pseudocomplete* [1] if X is quasi-regular and has a sequence $\{\mathfrak{P}_n : n \in \mathbb{N}\}$ of open π -bases such that $\bigcap_{n \in \mathbb{N}} V_n \neq \emptyset$, whenever $\overline{V_{n+1}} \subseteq V_n \in \mathfrak{P}_n$ for each $n \in \mathbb{N}$. Clearly, every almost countably subcompact space is pseudocomplete.

Question 3.11 ([7]). *If (X, d) is pseudocomplete (resp. subcompact, base-compact), must $(2^X, \mathcal{T}_{W(d)})$ be pseudocomplete (resp. subcompact, base-compact)?*

4. THE BAIRE PROPERTY OF WIJSMAN HYPERSPACES

Recall that a topological space X is said to be *Baire* if the intersection of every sequence of dense open subsets of X is still dense. Note that a closed subspace of a Baire space may not be Baire. A space is called *hereditarily Baire* if every non-empty closed subspace is Baire. For an alternative definition of a Baire space, refer to [19]. The third basic problem on the Wijsman topology concerns the Baire property of Wijsman hyperspaces.

Problem 4.1. *Let (X, d) be a metric space. Find characterizations for $(2^X, \mathcal{T}_{W(d)})$ to be a Baire space, in terms of some properties of (X, d) .*

Although it is not easy to find some completeness-type property for the Wijsman hyperspace of a completely metrizable metric space, the following positive result on the Baire property was discovered by Zsilinszky in 1996.

Theorem 4.2 ([30]). *Let (X, d) be a complete metric space. Then $(2^X, \mathcal{T}_{W(d)})$ is a Baire space.*

Note that, by Theorem 2.3 and Theorem 3.2, one should not expect that the conclusion of Theorem 4.2 can be strengthened to “strongly Baire”. However, this result can be strengthened as follows.

Theorem 4.3 ([5]). *Let X be a metrizable space. If X^{\aleph_0} is Baire, then $(2^X, \mathcal{T}_{W(d)})$ is Baire for every compatible metric d on X .*

In [11], Cao and Tomita constructed an example of a metric space (X, d) such that X^n is a Baire space for each $n \in \mathbb{N}$, but $(2^X, \mathcal{T}_V)$ is not a Baire space. Furthermore, Examples 3.3, 3.4, 3.7 or 3.8 imply that there exists a metric space (X, d) such that $(2^X, \mathcal{T}_{W(d)})$ is Baire, but $(2^X, \mathcal{T}_V)$ is not Baire. These examples suggest that the following question is interesting.

Question 4.4. *Let (X, d) be a metric space. If X^n is Baire for each $n \in \mathbb{N}$, must $(2^X, \mathcal{T}_{W(d)})$ be Baire?*

At the 10th Prague Topological Symposium in 2006, Zsilinszky posed the following five open questions in his talk.

Question 4.5. *If (X, d) is a Baire metric space, must $(2^X, \mathcal{T}_{W(d)})$ be Baire?*

Question 4.6. *If (X, d) is a hereditarily Baire metric space, must $(2^X, \mathcal{T}_{W(d)})$ be Baire?*

Question 4.7. *Let X be a metrizable space. If $(2^X, \mathcal{T}_{W(d)})$ is a Baire space for every compatible metric d on X , must X be Baire?*

Question 4.8. *Let X be a metrizable space. If $(2^X, \mathcal{T}_{W(d)})$ is a Baire space for every compatible metric d on X , must $(2^X, \tau_V)$ be Baire?*

Question 4.9. *Let X be a metrizable space. If $(2^X, \mathcal{T}_V)$ is a Baire space, must $(2^X, \mathcal{T}_{W(d)})$ be Baire for every compatible metric d on X ?*

Question 4.6 was completely solved by Cao and Tomita in [12], and its answer is affirmative. There are several partial affirmative answers to Question 4.5. Note that if a metrizable space X belongs to any of the following class of spaces:

- Baire spaces having a countable open π -base;
- separable Baire spaces;
- hereditarily Baire spaces;
- Baire spaces having a countable-in-itself open π -base;
- almost locally uK - U Baire spaces;
- almost locally separable Baire spaces;
- weakly α -favorable spaces;
- Čech-complete spaces;
- spaces with any of the (countable) Amsterdam properties;
- pseudocomplete spaces,

then X^{\aleph_0} is Baire, and thus by Theorem 4.3, the answer to Question 4.5 is affirmative. For details, refer to [5]. As mentioned at the end of [7], it is not possible to use the “barely Baire spaces” of Fleissner and Kunen in [18] as counterexamples to Question 4.5.

The answer to Question 4.7 is affirmative in the class of almost locally separable metrizable space, as the following result of Zsilinszky in [32] shows.

Theorem 4.10 ([32]). *Let X be an almost locally separable metrizable space. Then $(2^X, \mathcal{T}_{W(d)})$ is Baire for every compatible metric d on X if and only if X is Baire.*

Note that Theorem 4.10 also provides a solution to Problem 4.1 in the realm of almost locally separable metric spaces. However, the author does not know any information toward (partial) solutions to Questions 4.8 and 4.9.

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SCHOOL OF ENGINEERING, COMPUTER AND MATHEMATICAL SCIENCES, AUCKLAND UNIVERSITY
OF TECHNOLOGY, PRIVATE BAG 92006, AUCKLAND 1142, NEW ZEALAND
E-mail address: `jiling.cao@aut.ac.nz`