PRICING VARIANCE SWAPS IN A HYBRID MODEL OF STOCHASTIC VOLATILITY AND INTEREST RATE WITH REGIME-SWITCHING

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Abstract. In this paper, we consider the problem of pricing discretely-sampled variance swaps based on a hybrid model of stochastic volatility and stochastic interest rate with regime-switching. Our modeling framework extends the Heston stochastic volatility model by including the CIR stochastic interest rate and model parameters that switch according to a continuous-time observable Markov chain process. A semi-closed form pricing formula for variance swaps is derived. The pricing formula is assessed through numerical implementations, and the impact of including regime-switching on pricing variance swaps is also discussed.

1. Introduction

A *variance swap* is a forward contract on the future realized variance of returns of a specified asset. At maturity time $T > 0$, the variance swap rate can be evaluated as $V(T) = (RV - K) \times L$, where K is the annualized delivery or strike price for the swap, RV is the realized variance of the swap and L is the notional amount of the swap in dollars. A typical formula for measuring RV is

$$RV = \frac{AF}{N} \sum_{j=1}^{N} \left(\frac{S(t_j) - S(t_{j-1})}{S(t_{j-1})} \right)^2 \times 100^2, \tag{1}$$

where $S(t_j)$ is the closing price of the underlying asset at the j-th observation time t_j and N is the number of observations. The annualized factor AF follows the sampling frequency to convert the above evaluation to annualized variance points. Assuming there are 252 business days in a year, then AF is equal to 252 for daily sampling frequency. However, if the sampling frequency is monthly or weekly, then AF will be 12 or 52, respectively. The measure of realized variance requires monitoring the underlying price path discretely, usually at the end of a business day. For this purpose, we assume equally discrete observations to be compatible with the real market, which reduces to $AF = \frac{1}{\Delta t} = \frac{N}{T}$. The long position of variance swaps pays a fixed delivery price K at the expiration and receives the floating amounts of annualized realized variance, whereas the short position is the opposite.

Since variance swaps were first launched in 1998, the problem of how to price them has been an active research topic in mathematical and quantitative finance. Carr and Madan [2] combined static replication using options with dynamic trading in the futures to price and hedge variance swaps without specifying the volatility process. Demeterfi et al. [6] worked in the same direction by proving that a variance swap could be reproduced via a portfolio of standard options. A finite-difference method via dimension-reduction

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approach was explored in [12] to obtain high efficiency and accuracy for pricing discretely-sampled variance swaps. In [18, 19], Zhu and Lian extended the work in [12] by incorporating Heston two-factor stochastic volatility for pricing discretely-sampled variance swaps. However, a simpler approach was explored in [15], where the Schwarts solution procedure was applied to derive an affine solution of PDEs. Recently, to extend the work in [18] where stochastic interest rates were ignored, Cao et al. [4] employed a hybridization of the stochastic volatility model and the CIR interest rate model to investigate the pricing rates of variance swaps with discrete sampling. In [9], Elliott et al. proposed a continuous-time Markovian-regulated version of the Heston stochastic volatility model to distinguish different states of a business cycle. An analytical formula for pricing volatility swaps was obtained using the regime-switching Esscher transform and comparisons were made between models with and without switching regimes. The essence of incorporating regime-switching for pricing variance swaps under the Heston stochastic volatility model was illustrated in [8, 9], where a common assumption is "continuous sampling time". In fact, options of discretely sampled variance swaps were misvalued when the continuous sampling was used as an approximation, and large inaccuracies occurred in certain sampling periods, as discussed in [3, 8, 12, 19].

In the past decade, many researchers have considered to integrate Markovian regime-switching techniques with stochastic interest rate models. For example, in order to incorporate jumps and inconsistencies between different business stages, Elliott et al. [9] and Siu [17] used the regime-switching approach to extend the Cox-Ingersoll-Ross (CIR), the Hull-White and the Vasicek models respectively. However, there exists a gap in the literature regarding pricing volatility derivatives under stochastic volatility and stochastic interest rates with regime-switching. As far as we know, the only existing study was the one conducted in [16], which focused only on continuous sampling variance swaps and employed the PDE approach. In this paper, we address the issue of pricing discretely-sampled variance swaps under stochastic volatility and stochastic interest rate with regime-switching. We extend the framework of both [4] and [8] by incorporating the CIR stochastic interest rate into the Markov-modulated version of the Heston stochastic volatility model. This hybrid model possesses parameters that switch according to a continuous-time observable Markov chain process which can be interpreted as the states of an observable macroeconomic factor. Our approach is different from that of [16]. Instead of the continuous sampling approach, we use the discrete sampling approach to improve accuracy in pricing and computational efficiency.

The rest of this paper is organized as follows. In Section 2, a detailed description of regime-switching hybrid model is first provided, followed by derivation of the dynamics for the model under the T-forward measure. In Section 3, we derive the forward characteristic function in order to obtain the semi-analytical formula for the price of variance swaps. In Section 4, some numerical examples are given, demonstrating the accuracy of our solution and impacts of regime-switching. In Section 5, a brief summary and comparisons of our results with other relevant results in the literature are provided.

2. Modelling framework

In this section, we develop a hybrid model which combines the Heston stochastic volatility model with the one-factor CIR stochastic interest rate dynamics including regime-switching effects. A regime-switching model for pricing volatility derivatives was first considered by Elliot et al. [9]. Recently, Elliot and Lian [8] considered regime-switching effects on the Heston's stochastic volatility model. Our aim is to extend the work in [8] by incorporating stochastic interest rate into the modeling framework.

2.1. **The Heston-CIR model with regime-switching.** Let $\{S(t) : 0 \leq t \leq T\}$ be the process of certain asset price over a finite time horizon $[0, T]$. The Heston-CIR hybrid model is described by

$$\begin{cases} dS(t) = \mu S(t)dt + \sqrt{\nu(t)}S(t)dW_1(t), \quad 0 \leq t \leq T, \\ d\nu(t) = \kappa(\theta - \nu(t))dt + \sigma\sqrt{\nu(t)}dW_2(t), \quad 0 \leq t \leq T, \\ dr(t) = \alpha(\beta - r(t))dt + \eta\sqrt{r(t)}dW_3(t), \quad 0 \leq t \leq T, \end{cases} \tag{2}$$

where $\{\nu(t) : 0 \leq t \leq T\}$ is the stochastic instantaneous variance process and $\{r(t) : 0 \leq t \leq T\}$ is the process of stochastic instantaneous interest rate. The parameter κ determines the mean-reverting speed of $\nu(t)$, θ is its long-term mean and σ is its volatility. Similarly, α determines the speed of mean reversion for the interest rate process, β is the interest rate term structure and η controls the volatility of the interest rate. As mentioned in [5, 11], to ensure that the square root processes are always positive, it is required that $2\kappa\theta \geq \sigma^2$ and $2\alpha\beta \geq \eta^2$ respectively. Here, we assume that correlations involved in the above model are given by $(dW_1(t), dW_2(t)) = \rho dt$, $(dW_1(t), dW_3(t)) = 0$ and $(dW_2(t), dW_3(t)) = 0$, where ρ is a constant with $-1 \leq \rho \leq 1$. By the Girsanov theorem, there exists a risk-neutral measure $\mathbb{Q}$ equivalent to the real world measure $\mathbb{P}$ such that under $\mathbb{Q}$ system (2) is transformed into the form of

$$\begin{cases} dS(t) = r(t)S(t)dt + \sqrt{\nu(t)}S(t)d\widetilde{W}_1(t), \quad 0 \leq t \leq T, \\ d\nu(t) = \kappa^*(\theta^* - \nu(t))dt + \sigma\sqrt{\nu(t)}d\widetilde{W}_2(t), \quad 0 \leq t \leq T, \\ dr(t) = \alpha^*(\beta^* - r(t))dt + \eta\sqrt{r(t)}d\widetilde{W}_3(t), \quad 0 \leq t \leq T, \end{cases} \tag{3}$$

where $\kappa^* = \kappa + \lambda_1$, $\theta^* = \frac{\kappa\theta}{\kappa+\lambda_1}$, $\alpha^* = \alpha + \lambda_2$ and $\beta^* = \frac{\alpha\beta}{\alpha+\lambda_2}$ are the risk-neutral parameters, $\{\widetilde{W}_i(t) : 0 \leq t \leq T\}$ $(1 \leq i \leq 3)$ is a Brownian motion under $\mathbb{Q}$. Here, λ_j $(j = 1, 2)$ is the premium of volatility or interest rate risk.

The market dynamics is modelled by a continuous-time observable Markov chain $\mathbf{X} = \{X(t) : 0 \leq t \leq T\}$ with a finite state space $S = \{s_1, s_2, ..., s_N\}$. Without loss of generality, S can be identified with the set of unit vectors $\{e_1, e_2, ..., e_N\}$, where $e_i = (0, ..., 1, ..., 0)^\intercal \in \mathbb{R}^N$. An N-by-N rate matrix $Q = (q_{ij})_{1 \leq i,j \leq N}$ is used to generate the evolution of the chain under $\mathbb{Q}$. Here, $q_{ij} \geq 0$ for all $1 \leq i, j \leq N$ with $i \neq j$ and $\sum_{i=1}^N q_{ij} = 0$ for all $1 \leq j \leq N$. According to [10], a semi-martingale representation holds for the process $\mathbf{X}$ as follows

$$X(t) = X(0) + \int_0^t QX(s)ds + M(t), \tag{4}$$

where $\{M(t) : 0 \leq t \leq T\}$ is a $\mathbb{R}^N$-valued martingale with respect to the filtration generated by $\mathbf{X}$ under $\mathbb{Q}$. The regime-switching effect is captivated in our Heston-CIR model by assuming that the asset price, its volatility and the interest rate depend on market trends or other economic factors indicated by the regime-switching Markov chain $\mathbf{X}$. More precisely, the long-term mean of variance $\theta^*(t)$ of the asset price is given by $\theta^*(t) = \langle \theta^*, X(t) \rangle$, where $\theta^* = (\theta_1^*, \theta_2^*, ..., \theta_N^*)^\intercal$ with $\theta_i^* > 0$, for each $1 \leq i \leq N$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$. Similarly, the long-term mean of the interest rate $\beta^*(t)$ is given by $\beta^*(t) = \langle \beta^*, X(t) \rangle$, where $\beta^* = (\beta_1^*, \beta_2^*, ..., \beta_N^*)^\intercal$ with $\beta_i^* > 0$, for each $1 \leq i \leq N$. The Heston-CIR model under $\mathbb{Q}$ with regime switching is given by

$$\begin{cases} dS(t) = r(t)S(t)dt + \sqrt{\nu(t)}S(t)d\widetilde{W}_1(t), \quad 0 \leq t \leq T, \\ d\nu(t) = \kappa^*(\theta^*(t) - \nu(t))dt + \sigma\sqrt{\nu(t)}d\widetilde{W}_2(t), \quad 0 \leq t \leq T, \\ dr(t) = \alpha^*(\beta^*(t) - r(t))dt + \eta\sqrt{r(t)}d\widetilde{W}_3(t), \quad 0 \leq t \leq T. \end{cases} \tag{5}$$

Applying the Cholesky decomposition, we can re-write SDEs (5) as

$$\begin{pmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{pmatrix} = \mu^{\mathbb{Q}}dt + \Sigma \times C \times \begin{pmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{pmatrix}, \quad 0 \le t \le T, \tag{6}$$

with

$$\mu^{\mathbb{Q}} = \begin{pmatrix} r(t) \\ \kappa^*(\theta^*(t) - \nu(t)) \\ \alpha^*(\beta^*(t) - r(t)) \end{pmatrix}, \quad \Sigma = \begin{pmatrix} \sqrt{\nu(t)} & 0 & 0 \\ 0 & \sigma\sqrt{\nu(t)} & 0 \\ 0 & 0 & \eta\sqrt{r(t)} \end{pmatrix} \tag{7}$$

and

$$C = \begin{pmatrix} 1 & 0 & 0 \\ \rho & \sqrt{1-\rho^2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

such that

$$CC^{\intercal} = \begin{pmatrix} 1 & \rho & 0 \\ \rho & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and $dW_1^*(t)$, $dW_2^*(t)$ and $dW_3^*(t)$ are mutually independent under $\mathbb{Q}$ satisfying

$$\begin{pmatrix} d\widetilde{W}_1(t) \\ d\widetilde{W}_2(t) \\ d\widetilde{W}_3(t) \end{pmatrix} = C \times \begin{pmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{pmatrix}, \quad 0 \le t \le T.$$

2.2. **Model dynamics under T-forward measure.** In this subsection, we convert dynamics of the Heston-CIR model with regime-switching under $\mathbb{Q}$ to one under the T-forward measure $\mathbb{Q}^T$. To this end, we first derive a regime-switching exponential affine form for the price $P(t,T,r(t),X(t))$ of a zero-coupon bond under $\mathbb{Q}$.

Assume that the bond price $P(t,T,r(t),X(t))$ under $\mathbb{Q}$ has the following exponential affine form

$$P(t,T,r(t),X(t)) = e^{A(t,T,X(t)) - B(t,T)r(t)}, \tag{8}$$

where $A(t,T,X(t))$ and $B(t,T)$ are to be determined. The discounted bond price is given by

$$\widetilde{P}(t,T,r(t),X(t)) = e^{-\int_0^t r(s)ds} P(t,T,r(t),X(t)). \tag{9}$$

Applying Itô's formula to $\widetilde{P}(t,T,r(t),X(t))$ and noting that the non-martingale terms must sum up to zero, we obtain

$$\frac{\partial P}{\partial t} + \alpha^*(\beta^*(t) - r)\frac{\partial P}{\partial r} + \langle \mathbf{P}, QX(t)\rangle + \frac{1}{2}\frac{\partial^2 P}{\partial r^2}\eta^2 r - rP = 0, \tag{10}$$

with terminal condition $P(T,T,r(T),X(T)) = 1$, $\mathbf{P} = (P_1, P_2, ..., P_N)^{\intercal}$ and $P_i = P(t,T,r,e_i)$ for $1 \le i \le N$. Note that $X(t)$ takes one of the values from the set of unit vectors $\{e_1, e_2, ..., e_N\}$. If $X(t) = e_i$ for some $1 \le i \le N$, then

$$\begin{aligned} \theta^*(t) &= \langle \theta^*, X(t)\rangle = \theta_i^*, \\ \beta^*(t) &= \langle \beta^*, X(t)\rangle = \beta_i^*, \\ P(t,T,r(t),X(t)) &= P(t,T,r(t),e_i) = P_i. \end{aligned}$$

As a result, equation (10) becomes N coupled PDEs

$$\frac{\partial P_i}{\partial t} + \alpha^*(\beta_i^* - r)\frac{\partial P_i}{\partial r} + \langle \mathbf{P}, Qe_i\rangle + \frac{1}{2}\frac{\partial^2 P_i}{\partial r^2}\eta^2 r - rP_i = 0, \quad 1 \le i \le N \tag{11}$$

with terminal conditions $P_i(T,T,r(T))=1$. We then substitute the expressions of $\dfrac{\partial P}{\partial t}$, $\dfrac{\partial P}{\partial r}$ and $\dfrac{\partial^2 P}{\partial r^2}$ into the above PDEs to obtain the following ordinary differential equations as

$$\begin{cases} \dfrac{dB(t,T)}{dt} = \dfrac{1}{2}\eta^2 B(t,T)^2 + \alpha^* B(t,T) - 1, \\ \dfrac{dA_i}{dt} = \alpha^*\beta_i^* B(t,T) - e^{-A_i}\langle \widetilde{\mathbf{A}}, Qe_i\rangle, \quad 1\le i\le N, \end{cases} \tag{12}$$

where $A_i = A(t,T,e_i)$, $\widetilde{A}_i = e^{A_i}$ and $\widetilde{\mathbf{A}} = (\widetilde{A}_1, \widetilde{A}_2, ..., \widetilde{A}_N)^\intercal$, $1\le i\le N$. The terminal conditions become $B(T,T)=0$ and $A_i(T,T)=0$. Similar to the CIR model in [5], solution to the first equation of (12) is

$$B(t,T) = \frac{2\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)}{2\sqrt{(\alpha^*)^2+2\eta^2}+\left(\alpha^*+\sqrt{(\alpha^*)^2+2\eta^2}\right)\left(e^{(T-t)\sqrt{(\alpha^*)^2+2\eta^2}}-1\right)}.$$

To derive an expression for A_i's and $A(t,T,X(t))$, let $\Upsilon_i(t) = \alpha^*\beta_i^* B(t,T)$ for each $1\le i\le N$, and let $\mathrm{diag}(\Upsilon(t))$ denote the diagonal matrix whose entry on the i-th row and the i-column is $\Upsilon_i(t)$ for all $1\le i\le N$. Substituting $\widetilde{A}_i = e^{A_i}$ into (12), we can re-write the system of ordinary differential equations in (12) as the following matrix form

$$\frac{d\widetilde{\mathbf{A}}}{dt} = \left(\mathrm{diag}(\Upsilon(t)) - Q^\intercal\right)\widetilde{\mathbf{A}} \tag{13}$$

with $\widetilde{\mathbf{A}}(T,T)=\mathbf{1}$, where $\mathbf{1}=(1,1,...,1)^\intercal\in\mathbb{R}^N$. Let $\Phi(t)$ be the fundamental matrix of (13) with $\Phi(T)=I_N$, where I_N denotes the N-dimensional identity matrix. Then the solution to (13) with terminal condition $\widetilde{\mathbf{A}}(T,T)=\mathbf{1}$ can be expressed as $\widetilde{\mathbf{A}}(t,T) = \Phi(t)\mathbf{1}$. It follows that

$$\widetilde{A}_i(t,T) = \langle \Phi(t)\mathbf{1}, e_i\rangle \quad \text{and} \quad A(t,T,X(t)) = \ln(\langle \Phi(t)\mathbf{1}, X(t)\rangle).$$

Now, we implement the techniques of change of measure from $\mathbb{Q}$ to $\mathbb{Q}^T$. For brevity, let us denote the numeraire $e^{\int_0^t r(s)ds}$ by $N_{1,t}$ and the numeraire $P(t,T,r(t),X(t))$ by $N_{2,t}$. Then,

$$d\ln N_{1,t} = r(t)dt = \left(\int_0^t \alpha^*(\beta^* - r(s))ds\right)dt + \left(\int_0^t \eta\sqrt{r(s)}d\widetilde{W}_3(s)\right)dt.$$

So, the volatility for the numeraire $N_{1,t}$ is given by $\Sigma^{\mathbb{Q}} = (0,0,0)^\intercal$. Similarly, differentiating $\ln N_{2,t} = \ln\widetilde{A}(t,T,X(t)) - B(t,T)r(t)$ gives

$$\begin{aligned} d\ln N_{2,t} = \Bigg(& \frac{\dfrac{\partial \widetilde{A}(t,T,X(t))}{\partial t}}{\widetilde{A}(t,T,X(t))} - \frac{\partial B(t,T)}{\partial t}r(t) - B(t,T)\alpha^*(\beta^*(t) - r(t)) \\ & + \Big\langle \frac{\widetilde{\mathbf{A}}(t,T)}{\widetilde{A}(t,T,X(t))}, QX(t)\Big\rangle\Bigg)dt - B(t,T)\eta\sqrt{r(t)}d\widetilde{W}_3(t) \\ & + \Big\langle \frac{\widetilde{\mathbf{A}}(t,T)}{\widetilde{A}(t,T,X(t))}, dM(t)\Big\rangle. \end{aligned}$$

Note that $d\widetilde{W}_3(t)$ and $dM(t)$ are independent. So, the volatility for the numeraire $N_{2,t}$ is given by $\Sigma^T = \left(0,0,-B(t,T)\eta\sqrt{r(t)}\right)^\intercal$.

Using a formula in [1], we see that the drift μ^T of our SDEs under $\mathbb{Q}^T$ with regime-switching is given by

$$\mu^T = \mu^{\mathbb{Q}} - \left(\Sigma \times C \times C^{\intercal} \times (\Sigma^{\mathbb{Q}} - \Sigma^T)\right) = \begin{pmatrix} r(t) \\ \kappa^*(\theta^*(t) - \nu(t)) \\ \alpha^*\beta^*(t) - [\alpha^* + B(t,T)\eta^2]r(t) \end{pmatrix}$$

with Σ and $CC^{\intercal}$ as defined in (7). Therefore, the dynamics for (5) under $\mathbb{Q}^T$ is given by

$$\begin{pmatrix} \frac{dS(t)}{S(t)} \\ d\nu(t) \\ dr(t) \end{pmatrix} = \mu^T dt + \Sigma \times C \times \begin{pmatrix} dW_1^*(t) \\ dW_2^*(t) \\ dW_3^*(t) \end{pmatrix}. \tag{14}$$

In addition, under $\mathbb{Q}^T$, the semi-martingale decomposition of $\mathbf{X}$ is given by

$$X(t) = X(0) + \int_0^t Q^T(s)X(s)ds + M^T(t), \tag{15}$$

with the rate matrix $Q^T(t) = (q_{ij}^T(t))_{1\le i,j\le N}$ defined by

$${q_{ij}}^T(t) = \begin{cases} q_{ij}\dfrac{\widetilde{A}(t,T,e_j)}{\widetilde{A}(t,T,e_i)}, & i \neq j, \\ -\sum_{k\neq i} q_{ik}\dfrac{\widetilde{A}(t,T,e_k)}{\widetilde{A}(t,T,e_i)}, & i = j, \end{cases}$$

refer to [13] for details.

3. Derivation of pricing formula

In this section, we will derive a semi-closed form solution to the problem of pricing variance swaps under stochastic volatility and stochastic interest rate with regime-switching using characteristic functions. Let $y(T) = \ln S(T+\Delta) - \ln S(T)$. We have to evaluate the price conditional on the information about the sample path of $\mathbf{X}$ from $t = 0$ to $t = T + \Delta$. First, define $\mathscr{F}_1(t)$, $\mathscr{F}_2(t)$ and $\mathscr{F}_3(t)$ as the natural filtrations generated by $\{W_1^*(s) : 0 \le s \le t\}$, $\{W_2^*(s) : 0 \le s \le t\}$ and $\{W_3^*(s) : 0 \le s \le t\}$, respectively. Let $\mathscr{F}_X(t)$ be the filtration generated by $\{X(s) : 0 \le s \le t\}$. To obtain the characteristic function of $y(T)$, we need to evaluate the following conditional expectation in two steps:

$$\begin{aligned} &\mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(t)) \\ &= \mathbb{E}^T(\mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(T+\Delta)) \\ &\quad |\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(t)) \end{aligned} \tag{16}$$

In the first step, we compute $\mathbb{E}^T\left(e^{\phi y(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(T+\Delta)\right)$. In the second step, we compute $\mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(t))$.

3.1. Characteristic function for the given path $\mathscr{F}_X(T+\Delta)$.

We consider an enlarged filtration in which the forward characteristic function $f(\phi; t, T, \Delta, \nu(t), r(t))$ of $y(T)$ is defined by

$$f(\phi; t, T, \Delta, \nu(t), r(t)) = \mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(t)) \tag{17}$$

Proposition 1. If the underlying asset follows the dynamics (14), then

$$\begin{aligned} & f(\phi; t, T, \Delta, \nu(t), r(t)|\mathscr{F}_X(T+\Delta)) \\ = \;& e^{C(\phi,T)} j(D(\phi,T); t, T, \nu(t)) \cdot k(E(\phi,T); t, T, r(t)), \end{aligned}$$

where for any $0 \le t \le T$, $D(\phi,t)$, $j(\phi;t,T,\nu(t))$ and $k(\phi;t,T,r(t))$ are given by

$$\begin{aligned} &D(\phi,t) = \frac{a+b}{\sigma^2}\frac{1-e^{b(T+\Delta-t)}}{1-ge^{b(T+\Delta-t)}}, \\ &a = \kappa^* - \rho\sigma\phi, \quad b = \sqrt{a^2+\sigma^2(\phi-\phi^2)}, \quad g = \frac{a+b}{a-b}, \end{aligned} \tag{18}$$

with

$$\begin{aligned} &j(\phi;t,T,\nu(t)) = e^{F(\phi,t)+G(\phi,t)\nu(t)}, \quad F(\phi,t) = \textstyle\int_t^T \langle \kappa^*\theta^* G(\phi,s), X(s)\rangle ds, \\ &G(\phi,t) = \frac{2\kappa^*\phi}{\sigma^2\phi + (2\kappa^* - \sigma^2\phi)e^{\kappa^*(T-t)}}, \quad k(\phi;t,T,r(t)) = e^{L(\phi,t)+M(\phi,t)r(t)}, \end{aligned}$$

and $C(\phi,t)$, $E(\phi,t)$, $L(\phi,t)$ and $M(\phi,t)$ are determined by the following ODEs

$$\begin{cases} -\dfrac{dE}{dt} = \dfrac{1}{2}\eta^2 E^2 - (\alpha^* + B(t,T)\eta^2)E + \phi, \\ -\dfrac{dC}{dt} = \kappa^*\theta^*(t)D + \alpha^*\beta^*(t)E, \\ -\dfrac{dM}{dt} = \dfrac{1}{2}\eta^2 M^2 - (\alpha^* + B(t,T)\eta^2)M, \\ -\dfrac{dL}{dt} = \alpha^*\beta^*(t)M. \end{cases} \tag{19}$$

Proof. Here, we give a brief proof for Proposition 1. We represent the conditional forward characteristic function for $y(T)$ as

$$\begin{aligned} & f(\phi;t,T,\Delta,\nu(t),r(t)|\mathscr{F}_X(T+\Delta)) \\ = \;& \mathbb{E}^T(\mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(T)\vee\mathscr{F}_2(T)\vee\mathscr{F}_3(T)\vee\mathscr{F}_X(T+\Delta)) \\ & |\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(T+\Delta)). \end{aligned} \tag{20}$$

We first focus on calculating the inner expectation

$$\mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(T)\vee\mathscr{F}_2(T)\vee\mathscr{F}_3(T)\vee\mathscr{F}_X(T+\Delta)).$$

By defining function

$$U(\phi;t,\widetilde{s},\nu,r) = \mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(T+\Delta))$$

with $T \le t \le T+\Delta$, and applying the Feynman-Kac theorem, we obtain

$$\begin{aligned} &\frac{\partial U}{\partial t} + \frac{1}{2}\nu\frac{\partial^2 U}{\partial\widetilde{s}^2} + \frac{1}{2}\sigma^2\nu\frac{\partial^2 U}{\partial\nu^2} + \frac{1}{2}\eta^2 r\frac{\partial^2 U}{\partial r^2} + \rho\sigma\nu\frac{\partial^2 U}{\partial\widetilde{s}\partial\nu} + \left(r - \frac{1}{2}\nu\right)\frac{\partial U}{\partial\widetilde{s}} \\ &+ (\kappa^*(\theta^*(t)-\nu))\frac{\partial U}{\partial\nu} + \left(\alpha^*\beta^*(t) - (\alpha^* + B(t,T)\eta^2)r\right)\frac{\partial U}{\partial r} = 0, \\ &U(\phi;t=T+\Delta,\widetilde{s},\nu,r) = e^{\phi y(T)}, \end{aligned} \tag{21}$$

where $\widetilde{s}(t) = \ln S(t) - \ln S(T)$ in $T \le t \le T+\Delta$. In order to solve (21), we assume $U(\phi;t,\widetilde{s},\nu(t),r(t))$ in [11] has the following affine form

$$U(\phi;t,\widetilde{s},\nu(t),r(t)) = e^{C(\phi,t)+D(\phi,t)\nu+E(\phi,t)r+\phi\widetilde{s}}. \tag{22}$$

Substituting (22) into (21), we obtain the following three ODEs

$$\begin{cases} -\dfrac{dD}{dt} = \dfrac{1}{2}\phi(\phi-1) + (\rho\sigma\phi - \kappa^*)D + \dfrac{1}{2}\sigma^2 D^2, \\ -\dfrac{dE}{dt} = \dfrac{1}{2}\eta^2 E^2 - (\alpha^* + B(t,T)\eta^2)E + \phi, \\ -\dfrac{dC}{dt} = \kappa^*\theta^*(t)D + \alpha^*\beta^*(t)E, \end{cases} \tag{23}$$

with the initial conditions

$$C(\phi, T+\Delta) = 0, \quad D(\phi, T+\Delta) = 0, \quad E(\phi, T+\Delta) = 0. \tag{24}$$

Then, we can write the solution to the first ODE in (23) as

$$\begin{cases} D(\phi,t) = \dfrac{a+b}{\sigma^2}\dfrac{1-e^{b(T+\Delta-t)}}{1-ge^{b(T+\Delta-t)}}, \\ a = \kappa^* - \rho\sigma\phi, \quad b = \sqrt{a^2 + \sigma^2(\phi-\phi^2)}, \quad g = \dfrac{a+b}{a-b}. \end{cases} \tag{25}$$

Numerical integration is required to obtain the solutions of E and C.

Now, we move on to solve the outer expectation for $0 \le t \le T$. At $t = T$,

$$\begin{aligned} &\mathbb{E}^T(e^{\phi y(T)}|\mathscr{F}_1(T) \vee \mathscr{F}_2(T) \vee \mathscr{F}_3(T) \vee \mathscr{F}_X(T+\Delta)) \\ &= U(\phi; t = T, \widetilde{s}(T), \nu(T), r(T)) \\ &= e^{C(\phi,T) + D(\phi,T)\nu(T) + E(\phi,T)r(T)}. \end{aligned}$$

Define the following characteristic functions of $\nu(t)$ and $r(t)$, respectively

$$j(\phi; t, T, \nu(t)) = \mathbb{E}^T(e^{\phi\nu(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t)),$$

and

$$k(\phi; t, T, r(t)) = \mathbb{E}^T(e^{\phi r(T)}|\mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t)).$$

Then, we obtain the respective PDEs as

$$\begin{cases} \dfrac{\partial j}{\partial t} + \dfrac{1}{2}\sigma^2\nu\dfrac{\partial^2 j}{\partial \nu^2} + (\kappa^*(\theta^*(t) - \nu))\dfrac{\partial j}{\partial \nu} = 0, \\ j(\phi, t = T, T, \nu) = e^{\phi\nu}, \end{cases} \tag{26}$$

and

$$\begin{cases} \dfrac{\partial k}{\partial t} + \dfrac{1}{2}\eta^2 r\dfrac{\partial^2 k}{\partial r^2} + \left(\alpha^*\beta^*(t) - (\alpha^* + B(t,T)\eta^2)r\right)\dfrac{\partial k}{\partial r} = 0, \\ k(\phi, t = T, T, r) = e^{\phi r}. \end{cases} \tag{27}$$

Taking advantage of the affine-form solution techniques as those in [7, 11], we assume the solution to (26) is in the form of

$$j(\phi; t, T, \nu(t)) = e^{F(\phi,t) + G(\phi,t)\nu(t)}. \tag{28}$$

The functions $F(\phi, t)$ and $G(\phi, t)$ can be found by solving two ODEs

$$\begin{cases} -\dfrac{dG}{dt} = \dfrac{1}{2}\sigma^2 G^2 - \kappa^* G, \\ -\dfrac{dF}{dt} = \kappa^*\theta^*(t)G, \end{cases} \tag{29}$$

with the initial conditions

$$F(\phi, T) = 0, \qquad G(\phi, T) = \phi. \tag{30}$$

The solutions are

$$F(\phi,t)=\int_t^T \kappa^*\theta^*(s)G(\phi,s)ds,\quad G(\phi,t)=\frac{2\kappa^*\phi}{\sigma^2\phi+(2\kappa^*-\sigma^2\phi)e^{\kappa^*(T-t)}}.$$

Next, the function $k(\phi;t,T,r(t))=e^{L(\phi,T)+M(\phi,t)r(t)}$ is defined in order to derive a solution to (27). The initial conditions are $L(\phi,T)=0$ and $M(\phi,T)=\phi$. Then, L and M satisfy the following ODEs

$$\begin{cases} -\dfrac{dM}{dt}=\dfrac{1}{2}\eta^2M^2-(\alpha^*+B(t,T)\eta^2)M, \\ -\dfrac{dL}{dt}=\alpha^*\beta^*(t)M, \end{cases} \tag{31}$$

Combining the inner and outer expectation computations, we obtain the result claimed in the proposition. □

3.2. **Characteristic function for the given path $\mathscr{F}_X(t)$.** In this subsection, we derive a semi-closed formula for the characteristic function $f(\phi;t,T,\Delta,\nu(t),r(t))$. To achieve this, we need to evaluate the equation (18), where $\theta^*(t)$ and $\beta^*(t)$ depend on the path of the Markov chain process $\mathbf{X}$ up to $T+\Delta$,

$$\begin{aligned} & f(\phi;t,T,\Delta,\nu(t),r(t)) \\ = \;& \mathbb{E}^T(e^{C(\phi,T)}\cdot j(D(\phi,T);t,T,\nu(t))\cdot k(E(\phi,T);t,T,r(t)) \\ & |\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(t)) \\ = \;& \mathbb{E}^T\left(\exp\left(\int_T^{T+\Delta}\langle\alpha^*\beta^*E(\phi,s)+\kappa^*\theta^*D(\phi,s),X(s)\rangle ds\right.\right. \\ & +\int_t^T\langle\kappa^*\theta^*G(D(\phi,T),s),X(s)\rangle ds+\int_t^T\langle\alpha^*\beta^*M(E(\phi,T),s),X(s)\rangle ds \\ & +\frac{2\kappa^*D(\phi,T)}{\sigma^2D(\phi,T)+(2\kappa^*-\sigma^2D(\phi,T))e^{\kappa^*(T-t)}}\nu(t) \\ & \left.\left.+r(t)\int_t^T\left(\frac{1}{2}\eta^2M^2(E(\phi,T),s)-(\alpha^*+B(s,T)\eta^2)M(E(\phi,T),s)\right)ds\right)\right. \\ & \left.\Big|\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(t)\right) \\ = \;& \mathbb{E}^T\left(\exp(\int_t^{T+\Delta}\langle J(s),X(s)\rangle ds)|\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(t)\right) \\ & \times\exp\left(\nu(t)G(D(\phi,T),t)\right)\times\exp\left(r(t)M(E(\phi,T),t)\right). \end{aligned} \tag{32}$$

Here, the function $J(t)\in\mathbb{R}^N$ is given by

$$\begin{aligned} J(t)=&(\kappa^*\theta^*G(D(\phi,T),t)+\alpha^*\beta^*M(E(\phi,T),t))(1-H_T(t)) \\ &+(\alpha^*\beta^*E(\phi,t)+\kappa^*\theta^*D(\phi,t))H_T(t) \end{aligned} \tag{33}$$

along with $H_T(t)$ which is a Heaviside unit step function defined as

$$H_T(t)=\begin{cases} 1, & \text{if } t\geq T, \\ 0, & \text{else.} \end{cases}$$

Proposition 2. Let $\{X(t) : 0 \leq t \leq T\}$ be a regime-switching Markov chain with dynamics given by (15). Under $\mathbb{Q}^T$, $\exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right)$ is given by

$$\begin{aligned} &\mathbb{E}^T\left(\exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right)\Big|\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(t)\right) \\ &= \langle \Phi(t,T;J)X(t), \mathbf{1}\rangle, \end{aligned} \tag{34}$$

where the function $\Phi(t,T;J)$ is an N-by-N $\mathbb{R}$-valued matrix given by

$$\Phi(t,T;J) = \exp\left(\int_t^T (Q^T(s) + \mathrm{diag}(J(s)))ds\right), \tag{35}$$

with $\mathbf{1} = (1,1,...,1) \in \mathbb{R}^N$.

Proof. Consider $Z(t,T) = \exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right) X(T)$. Differentiating $Z(t,T)$ and using (15) yield

$$\begin{aligned} dZ(t,T) &= \exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right)(Q^T(T)X(T)dT + dM^T(T)) \\ &\quad + \langle J(T), X(T)\rangle \exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right) X(T)dT \\ &= \exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right) dM^T(T) + \langle J(T), X(T)\rangle Z(t,T)dT \\ &\quad + \exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right) Q^T(T)X(T)dT \\ &= \exp\left(\int_t^T \langle J(s), X(s)\rangle ds\right) dM^T(T) \\ &\quad + \left(Q^T(T) + \mathrm{diag}[J(T)]\right) Z(t,T)dT \end{aligned} \tag{36}$$

Integrating both sides of (36) gives

$$\begin{aligned} \int_t^T dZ(t,s) &= \int_t^T (Q^T(s) + \mathrm{diag}(J(s)))Z(t,s)ds \\ &\quad + \int_t^T \exp\left(\int_t^s \langle J(w), X(w)\rangle dw\right) dM^T(s). \end{aligned} \tag{37}$$

Put $\Psi(t,T;J) = \mathbb{E}^T\left[Z(t,T)\Big|\mathscr{F}_1(t)\vee\mathscr{F}_2(t)\vee\mathscr{F}_3(t)\vee\mathscr{F}_X(t)\right]$. Taking expectations in both sides of (37) results in

$$\Psi(t,T;J) = X(t) + \int_t^T (Q^T(s) + \mathrm{diag}(J(s)))\Psi(t,s;J)ds. \tag{38}$$

Suppose $\Phi(t,s;J)$ is the $N\times N$ matrix solution to the linear system of ordinary differential equation

$$\begin{cases} \dfrac{d\Phi(t,s;J)}{ds} = (Q^T(s) + \mathrm{diag}(J(s)))\Phi(t,s;J), \\ \Phi(t,t;J) = \mathrm{diag}(\mathbf{1}) = I_N. \end{cases} \tag{39}$$

Comparing with (38), we obtain the result $\Psi(t,T;J) = \Phi(t,T;J)X(t)$, which finally gives us formula (34). □

Now, substituting the result in Proposition 2 into (32) gives us the characteristic function of $y(T) = \ln S(T+\Delta) - \ln S(T)$ for the Heston-CIR model with regime-switching.

Proposition 3. If the underlying asset follows the dynamics (14), then the forward characteristic function of $y(T) = \ln S(T+\Delta) - \ln S(T)$ is given by

$$\begin{aligned} f(\phi; t, T, \Delta, \nu(t), r(t)) &= \mathbb{E}^T[e^{\phi y(T)} | \mathscr{F}_1(t) \vee \mathscr{F}_2(t) \vee \mathscr{F}_3(t) \vee \mathscr{F}_X(t)] \\ &= \exp(\nu(t) G(D(\phi, T), t)) \times \exp(r(t) M(E(\phi, T), t)) \\ &\quad \times \langle \Phi(t, T+\Delta; J) X(t), \mathbf{1} \rangle, \end{aligned} \tag{40}$$

where $D(\phi,t)$, $G(\phi,t)$, $J(t)$ and $\Phi(t, T+\Delta; J)$ are given by

$$\begin{aligned} &D(\phi,t) = \frac{a+b}{\sigma^2} \frac{1-e^{b(T+\Delta-t)}}{1-ge^{b(T+\Delta-t)}}, \quad a = \kappa^* - \rho\sigma\phi, \quad b = \sqrt{a^2 + \sigma^2(\phi - \phi^2)}, \\ &g = \frac{a+b}{a-b}, \quad G(\phi,t) = \frac{2\kappa^*\phi}{\sigma^2\phi + (2\kappa^* - \sigma^2\phi)e^{\kappa^*(T-t)}}, \\ &J(t) = (\kappa^*\theta^* G(D(\phi,T),t) + \alpha^*\beta^* M(E(\phi,T),t))\,(1 - H_T(t)) \\ &\qquad + (\alpha^*\beta^* E(\phi,t) + \kappa^*\theta^* D(\phi,t))\, H_T(t), \\ &\Phi(t, T+\Delta; J) = \exp\left(\int_t^{T+\Delta} (Q^T(s) + \mathrm{diag}(J(s)))ds \right), \end{aligned} \tag{41}$$

and $E(\phi,t)$ along with $M(\phi,t)$ are determined by the following ODEs

$$\begin{cases} -\dfrac{dE}{dt} = \dfrac{1}{2}\eta^2 E^2 - (\alpha^* + B(t,T)\eta^2)E + \phi, \\ -\dfrac{dM}{dt} = \dfrac{1}{2}\eta^2 M^2 - (\alpha^* + B(t,T)\eta^2)M. \end{cases} \tag{42}$$

Now, by using the valuation of the fair delivery price for a variance swap, and summarizing the whole previous procedure, we can write the forward characteristic function for a variance swap as

$$\begin{aligned} &\mathbb{E}^T\left(\left(\tfrac{S(t_j)}{S(t_{j-1})} - 1 \right)^2 \Big| \mathscr{F}_1(0) \vee \mathscr{F}_2(0) \vee \mathscr{F}_3(0) \vee \mathscr{F}_X(0) \right) \\ &= \mathbb{E}^T\left(e^{2y(t_{j-1})} - 2e^{y(t_{j-1})} + 1 | \mathscr{F}_1(0) \vee \mathscr{F}_2(0) \vee \mathscr{F}_3(0) \vee \mathscr{F}_X(0) \right) \\ &= f(2; 0, t_{j-1}, \Delta t, \nu(0), r(0)) - 2f(1; 0, t_{j-1}, \Delta t, \nu(0), r(0)) + 1, \end{aligned} \tag{43}$$

where $y(t_{j-1}) = \ln S(t_j) - \ln S(t_{j-1})$, $\Delta t = t_j - t_{j-1}$, and the characteristic function $f(\phi; t, T, \Delta, \nu(t), r(t))$ is given in equation (32). Hence, the fair strike price for a variance swap in terms of the spot variance $\nu(0)$ and the spot interest rate $r(0)$ under T-forward measure is given as

$$\begin{aligned} K &= \mathbb{E}^T(RV) \\ &= \frac{100^2}{T} \sum_{j=1}^{N} \left(f(2; 0, t_{j-1}, \Delta t, \nu(0), r(0)) - 2f(1; 0, t_{j-1}, \Delta t, \nu(0), r(0)) + 1 \right). \end{aligned} \tag{44}$$

4. Formula validation and results

In this section, we assess the performance of formula (44), by considering three regimes, denoted as $\{e_1, e_2, e_3\}$, representing the states *contraction*, *trough* and *expansion* of the business cycle, respectively. The contraction state can be defined as the situation when the economy starts slowing down, whereas the trough state happens when the economy hits bottom, usually in a recession. In addition, expansion is identified as the situation when the economy starts growing again. Here, we assume that the Heston-CIR model without regime-switching corresponds to the first regime and it will switch to the other

two regimes over time. Table 1 shows the set of parameters that we use to implement all the numerical experiments, unless otherwise stated.

TABLE 1. Model parameters of the Heston-CIR hybrid model with regime-switching.

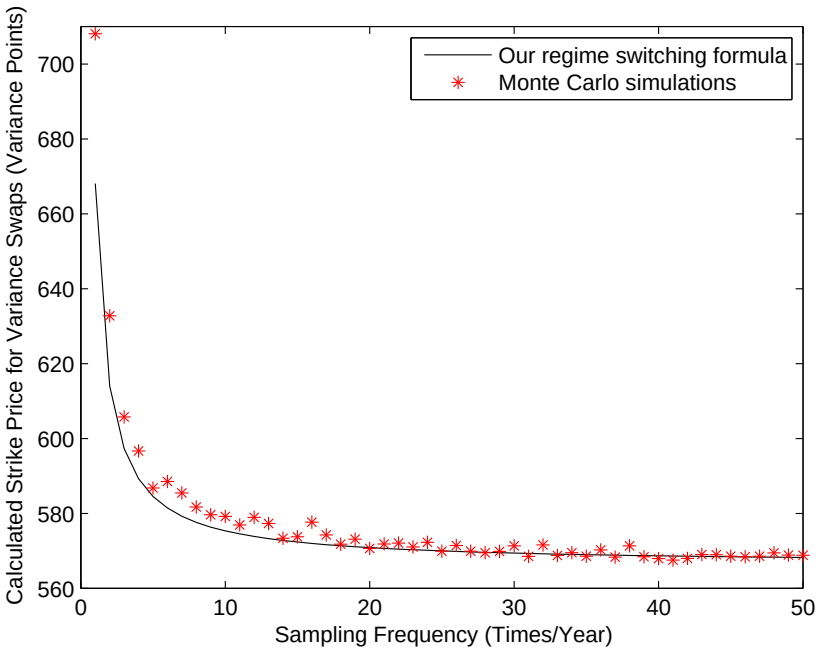
S_0	ρ	V_0	θ^*	κ^*	σ	r_0	α^*	β^*	η	T
1	-0.4	0.05	$(0.05, 0.075, 0.04)^\intercal$	2	0.1	0.05	1.2	$(0.05, 0.04, 0.075)^\intercal$	0.01	1

In addition, the rate matrix for the Markov chain $\mathbf{X}$ is given by

$$Q = \begin{pmatrix} -1 & 0.1 & 0.9 \\ 0.9 & -1 & 0.1 \\ 0.5 & 0.5 & -1 \end{pmatrix}.$$

4.1. **Validation of the pricing formula against Monte Carlo simulation.** We first demonstrate the validation of formula (44) against Monte Carlo simulation. Here, the sampling frequency varies from $N = 1$ up to $N = 52$, and the Monte Carlo simulation is conducted using the Euler discretization with 200,000 sample paths. The comparison is displayed in Figure 1.

FIGURE 1. Strike prices of variance swaps for the Heston-CIR model with regime-switching and the Monte Carlo simulation.

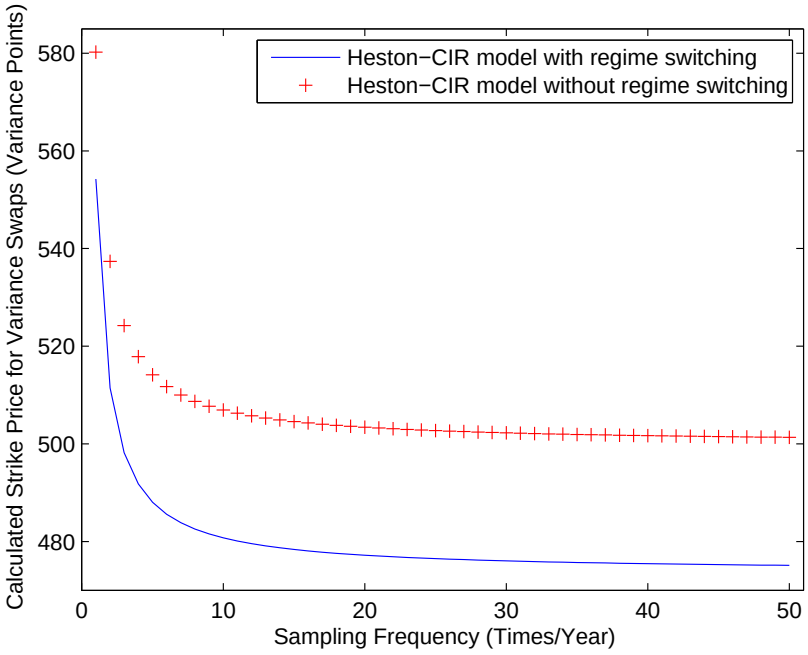


As shown in Figure 1, our pricing formula provides a satisfactory fit to the simulation for $N = 52$ which is the weekly sampling. In fact, the error calculated between our pricing formula and the simulation is less than 0.077% for $N = 52$, and this error will be reduced as the number of sample paths increases. In addition, it should be emphasized that for $N = 4$, the run time of our pricing formula is only 3.28 seconds, whereas the simulation takes about 8200 seconds. It is clear that our pricing formula attains almost

the same accuracy in far less time compared to the simulation which serves as benchmark values.

4.2. **Effect of regime-switching.** In order to explore the effect of regime-switching, in Figure 2, we present results produced by formula (44) and by the Heston-CIR model without regime-switching in [4]. For the Heston-CIR model without regime-switching, we fix the parameter values to be $\theta_1^* = 0.05$ and $\beta_1^* = 0.05$.

FIGURE 2. Strike prices of variance swaps for the Heston-CIR model with and without regime-switching.



We observe that the prices of variance swaps obtained from the Heston-CIR model with regime-switching are significantly lower than those from the corresponding model without regime-switching. For example, for $N = 52$, the difference between variance swaps prices calculated from the two models is 7.32%. This can be explained from the values of θ_1^* and β_1^* which remain constant, whereas the values of θ^* and β^* in the Heston-CIR model with regime-switching vary according to the changing states. Besides that, for the weekly sampling case, the difference in variance swaps prices between the two models becomes larger and stabilizes as the sampling frequency reaches 52. One possible explanation for this is the number of transitions between states in the Heston-CIR model with regime-switching increases as the sampling frequency increases.

In addition, we also examine the economic aftermath for the prices of variance swaps by allowing the Heston-CIR model to switch across three regimes. In particular, we denote $\theta_1^* = 0.05$ and $\beta_1^* = 0.05$ for the contraction state, $\theta_2^* = 0.075$ and $\beta_2^* = 0.04$ for the trough state, and $\theta_3^* = 0.04$ and $\beta_3^* = 0.075$ for the expansion state, respectively. These values are assumed by noting that a good (resp. bad) economy is identified by high (resp. low) interest rate and low (resp. high) volatility. We provide the variance swaps pricing outcome for these three regimes in Table 2.

TABLE 2. Comparison of variance swaps prices among different states in our pricing formula.

Sampling Frequency	State *Contraction*	State *Trough*	State *Expansion*
N=4	517.89	661.93	464.79
N=12	505.74	648.32	450.21
N=26	502.61	644.83	446.42
N=52	501.28	643.37	444.82

From Table 2, we discover that the the price of a variance swap is highest in the trough state, followed by the contraction state, and found lowest in the expansion state. This trend is consistent throughout all sampling frequencies from $N = 4$ to $N = 52$. We can relate this finding to the economic condition of each of the states. In particular, the trough state is the state with the worst economy among the three, whereas the expansion state resembles the best economy. Thus, the price of a variance swap is cheapest in the best economy among the three, and most expensive in the worst economy among all. This implies that regime-switching has an important impact in capturing the economic changes on the prices of variance swaps.

5. Conclusion

The evaluation of variance swaps has been an active research topic in recent years. In [16], the continuously sampled variance swaps were priced under the regime switching Schöbel-Zhu-Hull-White hybrid model. However, variance swaps are written on the realized variance based on daily closing prices in practice. To improve the pricing accuracy of these contracts, Zhu and Lian [18, 19] developed closed-form pricing formulas of discretely sampled variance swaps based on the framework of Heston's stochastic volatility model where the interest rate followed a deterministic process. In [4], a hybridization of the Heston stochastic volatility model and the CIR stochastic interest rate model was considered. The hybrid model extended the Heston stochastic volatility model in [18] by modelling the interest rate as the CIR process. The effect of stochastic interest rate on the price of discretely sampled variance swaps was demonstrated. Elliott and Lian [8] made another extension of the framework of Heston's stochastic volatility model in the direction of including regime switching dynamics in the model. It was shown that incorporating regime switching into the Heston model had a significant impact on the price of volatility swaps.

Since both regime switching and stochastic interest rate process affect the price of variance swaps, we propose a model incorporating both stochastic interest rate and regime switching effects. Specifically, the proposed model combines the CIR stochastic interest rate into the Markov-modulated regime switching version of the Heston stochastic volatility. Our model is capable of capturing several macroeconomic issues such as alternating business cycles. In particular, we assume that the long-term mean of variance of the risky stock and the long-term mean of the interest rate depend on the states of the economy indicated by a regime-switching Markov chain. We demonstrate our solution techniques and derive a semi-closed form formula for pricing variance swaps. Numerical experiments reveal that our pricing formula attains almost the same accuracy in far less time compared with the MC simulation. To analyse the effects of incorporating regime-switching into pricing variance swaps, we first compare the variance swaps prices calculated from the regime-switching Heston-CIR model with the corresponding model without regime-switching. We find that the prices of variance swaps obtained from the regime-switching Heston-CIR model are significantly different from those from the

Heston-CIR model without regime-switching. In our case, the Heston-CIR model without regime-switching corresponds to the state *contraction*, and the price of a variance swap obtained from the regime-switching Heston-CIR model is much lower than that obtained from the Heston-CIR model without regime-switching. If the Heston-CIR model without regime-switching corresponds to other states, the conclusion can be different. Next, we explore the economic consequence for the prices of variance swaps by allowing the Heston-CIR model to switch across three regimes defined as the best, moderate and worst economy. We notice that the price of a variance swap is cheapest in the best economy among the three, and most expensive in the worst economy among all. This confirms the essence of incorporating regime-switching in pricing variance swaps.

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