OSCILLATION REVISITED

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Abstract. In previous work by Beer and Levi [8, 9], the authors studied the oscillation $\Omega(f, A)$ of a function f between metric spaces $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ at a nonempty subset A of X, defined so that when $A = \{x\}$, we get $\Omega(f, \{x\}) = \omega(f, x)$, where $\omega(f, x)$ denotes the classical notion of oscillation of f at the point $x \in X$. The main purpose of this article is to formulate a general joint continuity result for $(f, A) \mapsto \Omega(f, A)$ valid for continuous functions.

1. Introduction

Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be two metric spaces, each with at least two points, and let $S_d(x, \varepsilon)$ denote the open d-ball of radius ε about $x \in X$. Suppose that f is a function from X to Y and $x \in X$ is arbitrary. Put

$$\omega_n(f, x) := \operatorname{diam}_\rho f\left(S_d\left(x, \frac{1}{n}\right)\right) \quad (n \in \mathbb{N}),$$

noting that the diameter of the image of the ball in the target space Y could be infinite. In any case, for each positive integer $n \in \mathbb{N}$,

$$\omega_n(f, x) \geq \omega_{n+1}(f, x),$$

so that

$$\lim_{n\to\infty} \omega_n(f, x) = \inf_{n\in\mathbb{N}} \omega_n(f, x)$$

is an extended nonnegative real number that is called the *oscillation* of f at x and is denoted by $\omega(f, x)$ (see, e.g., [20, p. 78]). Some basic well-known facts about oscillation are the following:

(1.1) $\omega(f, x) = 0$ if and only if f is continuous at x;
(1.2) $x \mapsto \omega(f, x)$ is upper semicontinuous;
(1.3) f is globally uniformly continuous if and only if $\langle \omega_n(f, \cdot)\rangle$ converges uniformly to the zero function on X.

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An oscillation function $\omega(f, \cdot)$ is thus a nonnegative extended real-valued upper semicontinuous function that must take on the value zero at each isolated point of the space, as each function is automatically continuous at isolated points. Conversely, a function g with these properties can be shown to be an isolation function for some Borel real-valued function defined on the space, as shown only fairly recently by Ewert and Ponomarev [17, Theorem 4].

One way to define the oscillation of a function f from X to Y at a nonempty subset A of X was proposed by Beer and Levi [8, 9]. Consistent with our notation for open balls, put

$$S_d(A, \varepsilon) := \bigcup_{a \in A} S_d(a, \varepsilon);$$

this union is often called the *ε-enlargement* of the set A, refer to [2]. Then for each $n \in \mathbb{N}$, we put

$$\Omega_n(f, A) := \sup \left\{ \rho(f(x), f(w)) : x, w \in S_d\left(A, \frac{1}{n}\right) \text{ and } d(x, w) < \frac{1}{n} \right\},$$

and call

$$\Omega(f, A) := \lim_{n \to \infty} \Omega_n(f, A) = \inf_{n \in \mathbb{N}} \Omega_n(f, A)$$

the *oscillation of f at A*. Easily, if $A = \{x\}$, then $\Omega(f, \{x\}) = \omega(f, x)$.

Concerning the oscillation $\Omega(f, A)$, one may ask the following question: What are the counterparts of properties (1.1) - (1.3)? First of all, note that neither continuity of f on X nor uniform continuity of f restricted to A ensures that $\Omega(f, A)$ is zero or even finite: consider $f : [0, \infty) \times [0, \infty) \to \mathbb{R}$ defined by $f(x, y) = xy$, where

$$A = \{(x, y) : x = 0 \text{ or } y = 0\}.$$

However, if A is compact and f is globally continuous, then the standard proof of the uniform continuity of f restricted to A shows that $\Omega(f, A) = 0$. More precisely, each nonempty subset A on which each globally continuous function on X has oscillation zero at A has this characteristic property [8, Theorem 5.2]: each sequence $\langle a_n \rangle$ in A along which $\lim_{n \to \infty} d(a_n, X - \{a_n\}) = 0$ must cluster. A subset that exhibits this property is called a *UC-subset*; trivially, each relatively compact subset is a UC-subset. If X is a UC-subset of itself, then the metric space is called a *UC-space*; their characteristic properties were first systematically described by Atsuji [1] (see also [2, 22]).

The family of all nonempty UC-subsets, like the family of all nonempty relatively compact subsets, form a *bornology*:

(1.4) they are an hereditary family;
(1.5) they are stable under finite unions;
(1.6) they form a cover of X.

The largest bornology on X is the family of all nonempty subsets $\mathcal{P}_0(X)$ and the smallest is the family of all nonempty finite subsets $\mathcal{F}_0(X)$. Three other bornologies of note are the family of all nonempty metrically bounded subsets, the family of all nonempty totally bounded subsets, and the family of all nonempty Bourbaki bounded subsets [5, 12, 18, 19, 28], also called the *finitely chainable subsets* [1].

Beer and Levi called f *strongly uniformly continuous* on A provided $\Omega(f, A) = 0$ as this property obviously implies that f restricted to A is uniformly continuous. They characterized strong uniform continuity in various ways, most notably, in terms of the preservation of nearness to subsets of A [8, Theorem 3.1], and in terms of the continuity of the induced direct image map from $\mathcal{P}_0(X)$ to $\mathcal{P}_0(Y)$ at points of $\mathcal{P}_0(A)$, where subsets of the domain and codomain are equipped with the the Hausdorff pseudometric topologies as determined by d and ρ, respectively [8, Theorem 3.3]. Strong uniform continuity of f at A is a variational alternative to the uniform continuity of the restriction of f to A: for every $\varepsilon > 0$, there exists $\delta > 0$ such that for each $a \in A$ and $x \in X, d(a, x) < \delta$ implies $\rho(f(a), f(x)) < \varepsilon$ [8, Theorem 4.3]. Furthermore, f is strongly uniformly continuous on A if and only if $\langle \omega_n(f, \cdot)\rangle$ converges uniformly to the zero function on A [8, Theorem 3.1], which presents rather convincing evidence that strong uniform continuity on a subset is the correct generalization of global uniform continuity.

With respect to the Hausdorff extended pseudometric topology τ_{H_d} determined by the Hausdorff distance H_d on $\mathcal{P}_0(X)$ [23, Definition 4.1.5], the map $A \mapsto \Omega(f, A)$ is upper semicontinuous for an arbitrary function f from X to Y [8, Theorem 4.3] which yields the known upper semicontinuity of $x \mapsto \omega(f, x)$ as a corollary, since $x \mapsto \{x\}$ is an isometric embedding of X into the hyperspace. Obviously, one cannot expect continuity of $A \mapsto \Omega(f, A)$ for an arbitrary function f with respect to any topology on $\mathcal{P}_0(X)$ with respect to which $x \mapsto \{x\}$ is a topological embedding. It is not even true that $A \mapsto \Omega(f, A)$ need be τ_{H_d}-continuous on $\mathcal{P}_0(X)$ for a continuous function f.

Example 1.1. For each $n \in \mathbb{N}$, let $A_n = \left\{\left(\frac{1}{n}, \frac{k}{n}\right) : k \in \mathbb{N}\right\}$, let $A = \{0\} \times [0, \infty)$ and let $X = A \cup \bigcup_{n\in\mathbb{N}} A_n$ equipped with the Euclidean metric d for the plane. Define $f : X \to \mathbb{R}$ by

$$f(x) = \begin{cases} 1, & \text{if } x = \left(\frac{1}{n}, \frac{k}{n}\right) \text{ with } k \geq n^2; \\ 0, & \text{otherwise.} \end{cases}$$

As f is zero on a neighborhood of each point of A and all other points of X are isolated, f is continuous on X. Clearly,

$$\lim_{n\to\infty} H_d(A, A_n) = 0, \quad \Omega(f, A_n) = 0,$$

because each A_n is a UC-subset, while $\Omega(f, A) = 1$. Thus, $\Omega(f, \cdot)$ fails to be continuous at A with respect to the H_d-pseudometric topology on $\mathcal{P}_0(X)$.

The last example notwithstanding, given a metrizable space X and a continuous real-valued function f on it, we can always find a compatible metric $\widehat{d}$ for which $A \mapsto \Omega(f, A)$ is $H_{\widehat{d}}$-continuous on $\mathcal{P}_0(X)$, in fact, identically equal to zero: let d be any compatible metric and put

$$\widehat{d}(x, w) = d(x, w) + |f(x) - f(w)|,$$

so that f is globally uniformly continuous (in fact Lipschitz) with respect to $\widehat{d}$. Still, one might look for stronger topologies on $\mathcal{P}_0(X)$ for a metrizable space X for which $A \mapsto \Omega(f, A)$ is continuous for all compatible metrics on X and for all continuous f with values in an arbitrary metric target space. We display such a topology here and use it to give a bona fide joint continuity of oscillation result. Finally, we show that a subset of X is a UC-subset with respect to a particular compatible metric on X as soon as $\Omega(f, A)$ is finite for all continuous real-valued functions f on X.

2. Preliminaries

All topological spaces will be assumed to contain at least two points. If X and Y are topological spaces, we write Y^X for the set of all functions from X to Y, and we denote the continuous functions from X to Y by $C(X,Y)$. We call an extended real-valued function defined on a topological space X *upper semicontinuous* (resp. *lower semicontinuous*) at $x \in X$ provided whenever $\langle x_\lambda \rangle_{\lambda \in \Lambda}$ is a net in X convergent to $x \in X$, we have $\limsup_{\lambda \in \Lambda}\ f(x_\lambda) \leq f(x)$ (resp. $\liminf_{\lambda \in \Lambda}\ f(x_\lambda) \geq f(x)$). Global upper semicontinuity means $\forall \alpha \in \mathbb{R},\ f^{-1}([-\infty, \alpha))$ is open, whereas global lower semicontinuity means $\forall \alpha \in \mathbb{R},\ f^{-1}((\alpha, \infty])$ is open. An extended real-valued function on a topological space X is continuous with respect to the usual topology on the target space $[-\infty, \infty]$ if and only if it is both lower semicontinuous and upper semicontinuous.

Let $\langle X, d \rangle$ be a metric space. For $A \in \mathcal{P}_0(X)$ and $x \in X$, we write $d(x, A)$ for $\inf\{d(x,a) : a \in A\}$, and $\mathrm{diam}_d(A)$ for

$$\sup\{d(a_1, a_2) : a_1 \in A \text{ and } a_2 \in A\}.$$

We now discuss some basic topologies on $\mathcal{P}_0(X)$ for a metrizable topological space X. In the literature, these topologies, called *hyperspace topologies*, are often restricted to the nonempty closed subsets of X. We restrict our attention to certain *admissible* hyperspace topologies, i.e., those for which $x \mapsto \{x\}$ is a topological embedding [2, p. 1]. First, if A and B are nonempty subsets of X, the *Hausdorff distance* between them as determined by a compatible metric d is defined by

$$H_d(A,B) := \inf\ \{\varepsilon > 0 : A \subseteq S_d(B, \varepsilon) \text{ and } B \subseteq S_d(A, \varepsilon)\}.$$

Clearly,

$$H_d(A,B) = H_d(\mathrm{cl}(A), \mathrm{cl}(B)),$$

and if d is unbounded, then we can find nonempty subsets A and B with $H_d(A,B) = \infty$. Hausdorff distance so defined gives an extended pseudometric on $\mathcal{P}_0(X)$. A countable local base for the topology τ_{H_d} that it determines at $A \in \mathcal{P}_0(X)$ consists of all sets of the form

$$\left\{ B \in \mathcal{P}_0(X) : H_d(A,B) < \frac{1}{n} \right\},$$

where n runs over the positive integers. It can be shown that $H_d(A,B)$ is the uniform distance between the associated distance functionals $d(\cdot, A)$ and $d(\cdot, B)$ [2, Theorem 1.5.1], and that two compatible metrics determined the same hyperspace topologies if and only if they are uniformly equivalent [2, Theorem 3.3.2].

We next introduce two "hit-and-miss" topologies on $\mathcal{P}_0(X)$, for which we need some additional (now standard) notation. For $E \in \mathcal{P}_0(X)$, we put

$$E^+ := \{A \in \mathcal{P}_0(X) : A \subseteq E\},$$

and for $\mathcal{E} \subseteq \mathcal{P}_0(X)$, we put

$$\mathcal{E}^- := \{A \in \mathcal{P}_0(X) : \forall E \in \mathcal{E},\ E \cap A \neq \emptyset\}.$$

The *finite topology* τ_{fin} on $\mathcal{P}_0(X)$, often called the *Vietoris topology*, is generated by all sets of the form V^+ where V is an open subset of X plus all sets of the form $\mathcal{V}^-$ where $\mathcal{V}$ is a finite family of open subsets of X (see, e.g., [2, 23, 27]). Replacing finite families of open sets by the larger collection of locally finite families of open sets, we obtain the finer *locally finite topology* τ_{locfin} [2, 6, 10, 25]. For nets of nonempty closed subsets, we cite the following classical results: $\langle A_\lambda \rangle$ converges in the finite (resp. locally finite) topology to A if and only if $\langle d(\cdot, A_\lambda) \rangle$ converges pointwise (resp. uniformly) to $d(\cdot, A)$ for each metric d compatible with the topology of X [2, 6, 7]. As noted above, uniform convergence of distance functionals with respect to a particular metric d means H_d-convergence of the underlying net of subsets; pointwise converge of distance functionals with respect to d is called *d-Wijsman convergence* for the underlying net of subsets (see, e.g., [3, 13, 16, 24, 29]).

3. Continuity of Oscillation with respect to τ_{locfin}

We first give an alternate presentation of the oscillation of a function f between metric spaces at a nonempty subset that we will use in the sequel.

Proposition 3.1. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $f \in Y^X$. Then for each $A \in \mathcal{P}_0(X)$, we have*

$$\Omega(f, A) = \inf_{n\in\mathbb{N}} \sup_{a\in A} \omega_n(f, a).$$

Proof. Put

$$\Omega^*_n(f, A) := \sup_{a\in A} \omega_n(f, a)$$

and then put

$$\Omega^*(f, A) := \inf_{n\in\mathbb{N}} \Omega^*_n(f, A).$$

We must show that

(3.1) $\Omega(f, A) \leq \Omega^*(f, A)$; and
(3.2) $\Omega^*(f, A) \leq \Omega(f, A)$.

In (3.1) we may assume that $\Omega^*(f, A)$ is finite, and in (3.2), we may assume that $\Omega(f, A)$ is finite. For (3.1), suppose $\Omega^*(f, A) < \alpha < \infty$ and choose $n \in \mathbb{N}$ such that $\Omega^*_n(f, A) < \alpha$. We claim that $\Omega_{2n}(f, A) < \alpha$. Let x and w be arbitrary members of $S_d\left(A, \frac{1}{2n}\right)$ with $d(x, w) < \frac{1}{2n}$. Choosing $a \in A$ with $d(x, a) < \frac{1}{2n}$, we have $\{x, w\} \subseteq S_d\left(a, \frac{1}{n}\right)$, and so

$$\rho(f(x), f(w)) \leq \operatorname{diam}_\rho f\left(S_d\left(a, \frac{1}{n}\right)\right) \leq \Omega^*_n(f, A),$$

so that

$$\Omega_{2n}(f, A) \leq \Omega^*_n(f, A) < \alpha,$$

which establishes the claim. This yields that $\Omega(f, A) \leq \Omega^*(f, A)$.

For (3.2), let α satisfy $\Omega(f, A) < \alpha < \infty$ and then choose $n \in \mathbb{N}$ such that $\Omega_n(f, A) < \alpha$. Let $a \in A$ be arbitrary and choose x, w in $S_d\left(a, \frac{1}{2n}\right)$. The triangle inequality gives

$$d(x, w) < 2 \cdot \frac{1}{2n} = \frac{1}{n},$$

and since $\{x, w\} \subseteq S_d\left(A, \frac{1}{n}\right)$, we have $\rho(f(x), f(w)) \leq \Omega_n(f, A)$. This yields $\omega_{2n}(f, a) \leq \Omega_n(f, A)$ and so

$$\Omega^*(f, A) \leq \Omega^*_{2n}(f, A) \leq \Omega_n(f, A) < \alpha,$$

from which $\Omega^*(f, A) \leq \Omega(f, A)$ follows. $\square$

As an application of our last result, we now provide a counterpart of (1.3) of Section 1 for the sequence $\langle \Omega_n(f, \cdot)\rangle$.

Proposition 3.2. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces, and f be a function from X to Y. Then f is globally uniformly continuous on X if and only if $\langle \Omega_n(f, \cdot)\rangle$ converges uniformly to the zero function on $\mathcal{P}_0(X)$.*

Proof. Suppose that f is globally uniformly continuous on X. Let $\varepsilon > 0$ be arbitrary. Then there exists $\delta > 0$ such that for any $x, y \in X$ with $d(x,y) < \delta$, $\rho(f(x), f(y)) < \varepsilon$. If we choose $n_0 \in \mathbb{N}$ such that $\frac{2}{n_0} < \delta$, then we have $\omega_n(f,x) \leq \varepsilon$ for all $x \in X$ whenever $n \geq n_0$. It follows from Proposition 3.1 that for any $A \in \mathcal{P}_0(X)$,

$$\Omega_{2n}(f,A) \leq \Omega_n^*(f,A) \leq \varepsilon$$

whenever $n \geq n_0$. This means that $\langle \Omega_n(f,\cdot) \rangle$ converges uniformly to the zero function on $\mathcal{P}_0(X)$.

To see the converse, assume that $\langle \Omega_n(f,\cdot) \rangle$ converges uniformly to the zero function on $\mathcal{P}_0(X)$. Let $\varepsilon > 0$ be arbitrary. Then, by Proposition 3.1, one can find some $n_0 \in \mathbb{N}$ such that for any $A \in \mathcal{P}_0(X)$,

$$\Omega_{2n}^*(f,A) \leq \Omega_n(f,A) < \varepsilon$$

whenever $n \geq n_0$. Thus, for any $x \in X$, we have

$$\omega_{2n}(f,x) = \Omega_{2n}^*(f,\{x\}) < \varepsilon,$$

whenever $n \geq n_0$. It follows that for any $x, y \in X$ with $d(x,y) < \frac{1}{2n_0}$, we have $\rho(f(x), f(y)) < \varepsilon$. This confirms the global uniform continuity of f on X. $\square$

In [3], it is shown that if $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ are metric spaces and $f \in Y^X$, then $A \mapsto \Omega(f,A)$ is upper semicontinuous on $\mathcal{P}_0(X)$ if it is equipped with τ_{H_d}. But the argument provided shows that something else is true, namely:

Proposition 3.3. *Let X be a metrizable space and let $\langle Y, \rho\rangle$ be a metric space. Then for each metric d on X that is compatible with the topology and for each $f \in Y^X$, the assignment $A \mapsto \Omega(f,A)$ computed using d and ρ is upper semicontinuous on $\mathcal{P}_0(X)$ equipped with τ_{fin}.*

Proof. Suppose $A \in \mathcal{P}_0(X)$. There is nothing to prove if $\Omega(f,A) = \infty$. Otherwise, take $\alpha \in \mathbb{R}$ with $\Omega(f,A) < \alpha$ as computed with respect to the compatible metric d. Choose $n \in \mathbb{N}$ with $\Omega_n(f,A) < \alpha$. Let $B \in S_d\left(A, \frac{1}{2n}\right)^+$ be arbitrary; by definition, $B \subseteq S_d\left(A, \frac{1}{2n}\right)$, so we have

$$\Omega(f,B) \leq \Omega_{2n}(f,B) \leq \Omega_n(f,A) < \alpha.$$

The result now follows because $S_d\left(A, \frac{1}{2n}\right)^+$ is a τ_{fin}-neighborhood of A. $\square$

Unfortunately, continuity of oscillation need not hold with respect to the finite topology for each compatible metric on the domain, even if the function f is real-valued and globally continuous. We provide a general construction in our next example.

Example 3.4. Suppose that we have a metric space $\langle X, d\rangle$ that contains a nonempty subset A that fails to be a UC subset but that is nevertheless a countable union of nonempty UC-subsets (for example, any unbounded dense-in-itself subset of $\mathbb{R}$ equipped with the usual metric does the job). Since the nonempty UC-subsets form a bornology, we can assume that there is an increasing sequence of nonempty UC-subsets $\langle A_n\rangle$ with union A. Clearly, $\langle A_n\rangle$ is τ_{fin}-convergent to A, but taking $f \in C(X,\mathbb{R})$, say, that fails to be strongly uniformly continuous at A (see [8, Corollary 5.3]), we have $\Omega(f,A_n) = 0$ for each n while $\Omega(f,A) > 0$.

Theorem 3.5. *Let X be a metrizable space and let $\langle Y,\rho\rangle$ be a metric space. Then for each $f \in C(X,Y)$ and for each compatible metric d for X, $A \mapsto \Omega(f,A)$ computed with respect to d and ρ is continuous on $\mathcal{P}_0(X)$ equipped with the locally finite topology.*

Proof. Fix a metric d compatible with the topology of X and let $f \in C(X,Y)$. We have already seen in our last result that $A \mapsto \Omega(f,A)$ is upper semicontinuous when $\mathcal{P}_0(X)$ is equipped with the weaker finite topology, so it remains to show the lower semicontinuity with respect to the locally finite topology.

Suppose $\Omega(f,A) > \alpha \in \mathbb{R}$. If $\Omega(f,A) = 0$, then $\mathcal{P}_0(X)$ is a neighborhood of A on which $\Omega(f,\cdot)$ exceeds α. Otherwise, without loss of generality, we may assume that $0 < \alpha < \Omega(f,A)$. We intend to produce a locally finite family $\mathcal{V}$ of open subsets of X such that $A \in \mathcal{V}^-$, and whenever $B \in \mathcal{V}^-$, we have $\Omega(f,B) > \alpha$.

Select $\beta \in (\alpha, \Omega(f,A))$ and set $n_1 = 2$. By Proposition 3.1, we can find $a_1 \in A$ with

$$\operatorname{diam}_\rho f\left(S_d\left(a_1, \frac{1}{n_1}\right)\right) > \beta.$$

By continuity of f at a_1, we can find an even integer $n_2 > n_1$ such that

$$\operatorname{diam}_\rho f\left(S_d\left(a_1, \frac{1}{n_2}\right)\right) < \beta.$$

Again by Proposition 3.1, choose $a_2 \in A$ with

$$\operatorname{diam}_\rho f\left(S_d\left(a_2, \frac{1}{n_2}\right)\right) > \beta.$$

Clearly, $a_2 \neq a_1$, and if $x \in S_d\left(a_1, \frac{1}{n_1}\right)$ then $S_d\left(x, \frac{2}{n_1}\right) \supseteq S_d\left(a_1, \frac{1}{n_1}\right)$, for whenever $w \in S_d\left(a_1, \frac{1}{n_1}\right)$, we have

$$d(w,x) \leq d(w,a_1) + d(a_1,x) < \frac{1}{n_1} + \frac{1}{n_2} < \frac{2}{n_1}.$$

Suppose we have chosen $2 = n_1 < n_2 < \cdots < n_k$ all even and distinct points $a_1, a_2, \ldots, a_k$ in A such that

(3.3) $\operatorname{diam}_\rho f\left(S_d\left(a_j, \frac{1}{n_j}\right)\right) > \beta$ for $j = 1,2,\ldots,k$;

(3.4) $\operatorname{diam}_\rho f\left(S_d\left(a_j, \frac{1}{n_{j+1}}\right)\right) < \beta$ for $j = 1,2,\ldots,k-1$;

(3.5) for each $x \in S_d\left(a_j, \frac{1}{n_{j+1}}\right)$, $S_d\left(x, \frac{2}{n_j}\right) \supseteq S_d\left(a_j, \frac{1}{n_j}\right)$ for $j = 1,2,\ldots,k-1$.

By continuity of f at a_k, we can choose an even integer $n_{k+1} > n_k$ satisfying

$$\operatorname{diam}_\rho f\left(S_d\left(a_k, \frac{1}{n_{k+1}}\right)\right) < \beta.$$

By Proposition 3.1, choose $a_{k+1} \in A$ such that

$$\operatorname{diam}_\rho f(S_d(a_{k+1}, \frac{1}{n_{k+1}})) > \beta.$$

By condition (3.4), $a_1, a_2, \ldots, a_k, a_{k+1}$ are all distinct, and an easy calculation shows that for each $x \in S_d\left(a_k, \frac{1}{n_{k+1}}\right)$, we have $S_d\left(x, \frac{2}{n_k}\right) \supseteq S_d\left(a_k, \frac{1}{n_k}\right)$.

Continuing to produce a strictly increasing sequence of even integers $\langle n_k \rangle$ and a sequence of distinct points $\langle a_k \rangle$ in A, we conclude that the family of balls

$$\left\{ S_d\left(a_k, \frac{1}{n_k}\right) : k \in \mathbb{N} \right\}$$

is locally finite, else $\langle a_k \rangle$ would have a cluster point at which continuity of f must fail by condition (3.3). Thus, $\{S_d\left(a_k, \frac{1}{n_{k+1}}\right) : k \in \mathbb{N}\}$, being a family of smaller balls, is also locally finite and $A \in \{S_d\left(a_k, \frac{1}{n_{k+1}}\right) : k \in \mathbb{N}\}^-$. We intend to show that if $B \in \mathcal{P}_0(X)$ hits each ball $S_d\left(a_k, \frac{1}{n_{k+1}}\right)$, then $\Omega(f, B) > \alpha$. To see this, it suffices by Proposition 3.1 to produce for each $m \in \mathbb{N}$ some $b \in B$ with

$$\mathrm{diam}_\rho f\left(S_d\left(b, \frac{1}{m}\right)\right) > \beta.$$

Choose n_j with $\frac{2}{n_j} < \frac{1}{m}$ and then $b \in B \cap S_d\left(a_j, \frac{1}{n_{j+1}}\right)$. Using condition (3.3) and condition (3.5),

$$f\left(S_d\left(b, \frac{1}{m}\right)\right) \supseteq f\left(S_d\left(b, \frac{2}{n_j}\right)\right) \supseteq f\left(S_d\left(a_j, \frac{1}{n_j}\right)\right)$$

so that by condition (3.3) and Proposition 3.1, $\Omega(f, B) \geq \beta > \alpha$. $\square$

4. Joint Continuity of Oscillation

In [8], Beer and Levi introduced the variational notion of strong uniform convergence of a net of functions $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ from $\langle X, d \rangle$ to $\langle Y, \rho \rangle$ to a function f on a nonempty subset A of X: for each $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ such that for each $\lambda \succeq \lambda_0$, there exists $\delta > 0$ such that for all $x \in S_d(A, \delta)$, $\rho(f_\lambda(x), f(x)) < \varepsilon$ (notice that δ can depend on λ!). The family of nonempty subsets on which strong uniform convergence occurs is stable under finite unions and is hereditary, and strong uniform convergence on (each member of) a bornology is compatible with a uniformizable topology on Y^X. If each $f_\lambda \in C(X, Y)$ and $f \in C(X, Y)$ and $\langle f_\lambda \rangle_{\lambda \in \Lambda}$ is pointwise convergent to f, then strong uniform convergence must occur on each singleton subset of X. Conversely, strong uniform convergence of a net of continuous functions on each singleton to $f \in Y^X$ ensures that f is continuous [8, 11]. One cannot overstate how well strong uniform convergence on a bornology $\mathcal{B}$ comports with strong uniform continuity of functions on $\mathcal{B}$. A number of subsequent papers on this convergence notion/topology have been written [9, 14, 15, 21]; in particular, nontransparent necessary and sufficient conditions on a bornology $\mathcal{B}$ for the topology of strong uniform convergence to collapse to the classical topology of $\mathcal{B}$-uniform convergence on $C(X, Y)$ [26] are known [9].

But one result that we would hope for is not be had: joint upper semicontinuity of $(f, A) \mapsto \Omega(f, A)$ need not hold even if we restrict our functions to those that are strongly uniformly continuous on a bornology $\mathcal{B}$ and restrict our sets to members of $\mathcal{B}$, where our functions are topologized by the topology of strong uniform convergence on the bornology and $\mathcal{B}$ is topologized by Hausdorff distance [8, Example 6.13]. What is needed is a somewhat stronger convergence notion for our functions that is obtained by flipping quantifiers in the definition of strong uniform convergence.

Definition 4.1. Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces, and let A be a nonempty subset of X. A net of functions $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ from X to Y is declared *very strongly uniformly convergent to* $f : X \to Y$ *on* A if for each $\varepsilon > 0$ there exists $\lambda_0 \in \Lambda$ and $\delta > 0$ such that for all $\lambda \succeq \lambda_0$ and all $x \in S_d(A, \delta)$, $\rho(f_\lambda(x), f(x)) < \varepsilon$.

To see the difference between strong uniform convergence and very strong uniform convergence for continuous functions, let f_n be the piecewise linear function on $[0, 1]$ whose graph joins $(0, 0)$ to $(\frac{1}{2n}, 1)$ to $(\frac{1}{n}, 0)$ to $(1, 0)$, let f be the zero function on $[0, 1]$, and let $A = \{0\}$. Then, it is readily checked that $\langle f_n\rangle$ is strongly uniformly convergent to f on A, but $\langle f_n\rangle$ is not very strongly uniformly convergent to f on A.

While strong uniform convergence on each singleton of a pointwise convergent net of continuous functions, being equivalent to continuity of the limit, cannot ensure uniform convergence on compact subsets, we have the following result whose easy proof is left to the reader.

Proposition 4.2. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces and let $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ be a net in Y^X very strongly uniformly convergent to $f : X \to Y$ on each singleton subset of X. Then very strong uniform convergence on compact subsets to f occurs.*

Theorem 4.3. *Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be two metric spaces. Let $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ be a net in Y^X and $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ be a net of nonempty subsets of X, where Λ is a directed set. Suppose that $A \in \mathcal{P}_0(X)$ and $f : X \to Y$ satisfy*

(i) *for all $n \in \mathbb{N}, A_\lambda \subseteq S_d\left(A, \frac{1}{n}\right)$ eventually; and*
(ii) *$\langle f_\lambda\rangle_{\lambda\in\Lambda}$ is very strongly uniformly convergent to $f \in Y^X$ on A.*

Then

$$\Omega(f, A) \geq \limsup_{\lambda\in\Lambda}\ \Omega(f_\lambda, A_\lambda).$$

Proof. We may assume that $\Omega(f, A)$ is finite. Let $\beta > \Omega(f, A)$ be arbitrary and choose $\varepsilon > 0$ such that $\beta > \Omega(f, A) + 3\varepsilon$. Choose $n \in \mathbb{N}$ and $\lambda_0 \in \Lambda$ so large that

$$\Omega_n(f, A) < \Omega(f, A) + \varepsilon$$

and such that whenever $\lambda \succeq \lambda_0$ we have both

(4.1) $\sup\left\{\rho(f(x), f_\lambda(x)) : x \in S_d\left(A, \frac{1}{n}\right)\right\} < \varepsilon$;
(4.2) $A_\lambda \subseteq S_d\left(A, \frac{1}{2n}\right)$.

For $\lambda \succeq \lambda_0$, by (4.2) we have $S_d\left(A_\lambda, \frac{1}{2n}\right) \subseteq S_d\left(A, \frac{1}{n}\right)$, and so whenever $\{x, w\} \subseteq S_d\left(A_\lambda, \frac{1}{2n}\right)$ and $d(x, w) < \frac{1}{2n}$, we get

$$\begin{array}{rcl}\rho(f_\lambda(x), f_\lambda(w)) & \leq & \rho(f_\lambda(x), f(x)) + \rho(f(x), f(w)) + \rho(f(w), f_\lambda(w)) \\ & < & \varepsilon + \Omega_n(f, A) + \varepsilon < \Omega(f, A) + 3\varepsilon.\end{array}$$

From this, it follows that whenever $\lambda \succeq \lambda_0$,

$$\Omega(f_\lambda, A_\lambda) \leq \Omega_{2n}(f_\lambda, A_\lambda) \leq \Omega(f, A) + 3\varepsilon < \beta$$

so that

$$\Omega(f, A) \geq \limsup\nolimits_{\lambda\in\Lambda} \Omega(f_\lambda, A_\lambda),$$

and the proof is complete. $\square$

Theorem 4.4. *Let X be a metrizable space and $\langle Y, \rho\rangle$ be a metric space. Let $(\Lambda, \succeq)$ be a directed set and let $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ be a net in $C(X,Y)$ and let $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ be a net in $\mathcal{P}_0(X)$. Suppose that $\langle A_\lambda\rangle_{\lambda\in\Lambda}$ is convergent in the locally finite topology τ_{locfin} to $A \in \mathcal{P}_0(X)$, and suppose $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ is very strongly uniformly convergent to $f \in C(X,Y)$ on A. Then for each metric d compatible with the topology of X,*

$$\Omega(f,A) = \lim_{\lambda\in\Lambda}\ \Omega(f_\lambda, A_\lambda),$$

where oscillations are computed with respect to d and ρ.

Proof. Since containment in $S_d(A,\delta)$ means membership to $S_d(A,\delta)^+$ and $S_d(A,\delta)^+$ belongs to the locally finite topology τ_{locfin}, the last result gives

$$\limsup_{\lambda\in\Lambda}\ \Omega(f_\lambda, A_\lambda) \leq \Omega(f,A).$$

It remains to show that

$$\Omega(f,A) \leq \liminf_{\lambda\in\Lambda}\ \Omega(f_\lambda, A_\lambda).$$

We may assume that $\Omega(f,A) \neq 0$. Let $0 < \alpha < \Omega(f,A)$ be arbitrary, and choose $\varepsilon > 0$ with $\alpha + 3\varepsilon < \Omega(f,A)$. The proof of Theorem 3.5 shows that there exists a locally finite family $\mathcal{V}$ of nonempty open subsets of X (in fact open balls) such that $A \in \mathcal{V}^-$ and if $B \in \mathcal{V}^-$, then $\Omega(f,B) > \alpha + 3\varepsilon$. By very strong uniform convergence on A, choose $\delta > 0$ and $\lambda_1 \in \Lambda$ such that whenever $\lambda \succeq \lambda_1$,

$$\sup\ \{\rho(f(x), f_\lambda(x)) : x \in S_d(A,\delta)\} < \varepsilon.$$

There exists $\lambda_2 \succeq \lambda_1$ such that

$$A_\lambda \in \mathcal{V}^- \cap S_d\left(A, \frac{\delta}{2}\right)^+$$

for all $\lambda \succeq \lambda_2$. Let n be any positive integer such that $\frac{1}{n} < \frac{\delta}{2}$. Whenever $\{x,w\} \subseteq S_d\left(A_\lambda, \frac{1}{n}\right)$ and $d(x,w) < \frac{1}{n}$, we have $\{x,w\} \subseteq S_d(A,\delta)$, and the triangle inequality yields for all $\lambda \succeq \lambda_2$,

$$\rho(f_\lambda(x), f_\lambda(w)) > \rho(f(x), f(w)) - 2\varepsilon.$$

It follows that for all n sufficiently large, we get

$$\Omega_n(f_\lambda, A_\lambda) \geq \Omega_n(f, A_\lambda) - 2\varepsilon \geq \Omega(f, A_\lambda) - 2\varepsilon > \alpha + \varepsilon,$$

and so

$$\Omega(f_\lambda, A_\lambda) \geq \alpha + \varepsilon > \alpha$$

for all $\lambda \succeq \lambda_2$. We may now conclude that

$$\Omega(f,A) \leq \liminf_{\lambda\in\Lambda}\ \Omega(f_\lambda, A_\lambda),$$

which completes the proof. $\square$

Corollary 4.5. *Let X be a metrizable space and $\langle Y, \rho\rangle$ be a metric space. Suppose $C(X,Y)$ is equipped with the topology of ρ-uniform convergence, and $\mathcal{P}_0(X)$ is equipped with the locally finite topology. Then for each metric d compatible with the topology of X, $(f,A) \to \Omega(f,A)$ is continuous on $C(X,Y) \times \mathcal{P}_0(X)$ equipped with the product topology, where oscillations are computed with respect to d and ρ.*

If we have very strong uniform convergence of a net of functions on each member of some family of nonempty subsets $\mathcal{A}$ of $\langle X, d\rangle$ then we evidently have very strong uniform convergence on subsets of members of $\mathcal{A}$ and on finite unions of members of $\mathcal{A}$. Thus, if $\mathcal{A}$ forms a cover of X, there is no loss of generality in assuming that $\mathcal{A}$ is a bornology.

Let $\langle X, d\rangle$ and $\langle Y, \rho\rangle$ be metric spaces, and let $\mathcal{B}$ be a bornology on X. We say that a net of functions $\langle f_\lambda\rangle_{\lambda\in\Lambda}$ from X to Y is *very strongly uniformly convergent to* $f : X \to Y$ *on* $\mathcal{B}$ if it is very strongly uniformly convergent to f on every member B of $\mathcal{B}$. This usage parallels the notion of strong uniform convergence in Y^X on $\mathcal{B}$ where the convergence is always compatible with a uniformizable topology [8]. It came as a surprise to the authors that very strong uniform convergence need not be topological even if we restrict our attention to $C(X, \mathbb{R})$ and our bornology is the bornology of nonempty finite subsets $\mathcal{F}_0(X)$, which corresponds to very strong uniform convergence on each singleton subset. We show that the iterated limit condition [23, p. 30] which is necessary for the convergence to be topological can fail for a sequence of sequences of real-valued continuous functions.

Example 4.6. Our base metric space $\langle X, d\rangle$ is the sequence space ℓ_∞, in which for each $n \in \mathbb{N}$, e_n is the sequence whose nth term is one and whose other terms are all zero. For our kth sequence in $C(\ell_\infty, \mathbb{R})$ we put

$$f_{k,n}(x) = \begin{cases} 1 - 3^k d(x, \frac{1}{k}e_n), & \text{if } d(x, \frac{1}{k}e_n) < \frac{1}{3^k}; \\ 0, & \text{otherwise.} \end{cases}$$

The sequence $f_{k,1}, f_{k,2}, \ldots, f_{k,n}, \ldots$ converges very strongly uniformly at each point of ℓ_∞ to the zero function as it is eventually zero in a fixed neighborhood of each point that does not depend on n. However, if we direct $\mathbb{N} \times \mathbb{N}^{\mathbb{N}}$ with the pointwise order, the net $(k, \phi) \mapsto f_{k,\phi(k)}$ evidently fails to converge very strongly uniformly at the origin of ℓ_∞ to the zero function.

5. A New Characteriation of UC Subsets

In the introduction, we described sequentially those nonempty subsets A of a metric space $\langle X, d\rangle$ on which each continuous function on X has oscillation zero at A; for a very different sequential description, the reader may consult [4, Theorem 3.5]. Such subsets, called UC-subsets, can also be described in terms of gaps, where the *gap* between two nonempty subsets of X is the infimum of the distances between pairs of points one in each set, see [2]. We call two nonintersecting sets *asymptotic* if the gap between them is zero.

For notational economy and following [8], we now write $I(x)$ for $d(x, X\backslash\{x\})$; the functional $I(\cdot)$ measures the isolation of points of X and $I(x) = 0$ means that x is a limit point of X.

Theorem 5.1. *Let A be a nonempty subset of $\langle X, d\rangle$. The following conditions are equivalent:*

(i) *A is a UC-subset;*
(ii) *for each $f \in C(X, \mathbb{R})$, $\Omega(f, A)$ is finite;*
(iii) *whenever C and E are nonempty closed subsets of X with $C \subseteq A$ and $C \cap E = \emptyset$, then the gap between C and E is positive.*

Proof. (i) $\Rightarrow$ (ii) trivially follows from [8, Theorem 5.2].

For (ii) $\Rightarrow$ (iii), if (iii) fails for some C and E, by the Tietze extension theorem, we can find $f \in C(X, \mathbb{R})$ mapping each point of C to zero and such that for each $e \in E$, $f(e) = d(e, C)^{-1}$. Now for each $n \in \mathbb{N}$, there exists $c_n \in C$ and $e_n \in E$ with $d(c_n, e_n) < \frac{1}{n}$ and of course $\{c_n, e_n\} \subseteq S_d\left(A, \frac{1}{n}\right)$. As a result $\Omega_n(f, A) > n$ and so $\Omega(f, A) = \infty$.

Only (iii) $\Rightarrow$ (i) remains. Suppose (i) fails; then we can find a sequence $\langle a_n \rangle$ in A that fails to cluster but for which $\lim_{n\to\infty} I(a_n) = 0$. By passing to a subsequence, we may assume all terms are distinct and either (a) all terms are limit points of X, or (b) all terms are isolated points of X. In case (a), choose a strictly increasing sequence of positive integers $\langle k_n \rangle$ such that for each $n \in \mathbb{N}$,

$$\frac{1}{k_n} < \frac{1}{3}\ d(a_n, \{a_j : j \neq n\});$$

then $\left\{S_d\left(a_n, \frac{1}{k_n}\right) : n \in \mathbb{N}\right\}$ is a pairwise disjoint family of balls. Choose $e_n \in S_d\left(a_n, \frac{1}{k_n}\right)$ different from a_n, and put $C := \{a_n : n \in \mathbb{N}\}$ and $E := \{e_n : n \in \mathbb{N}\}$. Since $\langle a_n \rangle$ can't cluster, neither can $\langle e_n \rangle$ as $\lim_{n\to\infty} d(a_n, e_n) = 0$. Thus, C and E are disjoint asymptotic closed subsets of X with $C \subseteq A$, which violates (iii).

Case (b) is a little more delicate, involving an iterative procedure. Put $n_1 = 1$ and choose $x_{n_1} \in X$ so that

$$0 < d(a_{n_1}, x_{n_1}) < I(a_{n_1}) + \frac{1}{n_1}.$$

Having chosen $n_1 < n_2 < \cdots < n_k$ and $x_{n_1}, x_{n_2}, \ldots, x_{n_k}$ not necessarily distinct in X such that $\{a_{n_1}, a_{n_2}, \ldots, a_{n_k}\}$ and $\{x_{n_1}, x_{n_2}, \ldots, x_{n_k}\}$ form disjoint sets, put

$$\delta = \min\left\{\{I(a_{n_j}) : j \leq k\}, \{I(x_{n_j}) : x_{n_j} \text{ is an isolated point of } X, j \leq k\}\right\}.$$

Since the measure of isolation functional goes to zero along the sequence $\langle a_n \rangle$, we can find $n_{k+1} > n_k$ and then $x_{n_{k+1}} \in X$ with

$$0 < d(a_{n_{k+1}}, x_{n_{k+1}}) < I(a_{n_{k+1}}) + \frac{1}{n_{k+1}} < \delta.$$

Since $I(x_{n_{k+1}}) < \delta$ as well, we obtain the disjointness of $\{a_{n_1}, a_{n_2}, \ldots, a_{n_{k+1}}\}$ and $\{x_{n_1}, x_{n_2}, \ldots, x_{n_{k+1}}\}$.

In this way we produce sequences $\langle a_{n_j} \rangle$ and $\langle x_{n_j} \rangle$ whose sets of terms do not overlap and such that for each $j \in \mathbb{N}$,

$$0 < d(a_{n_j}, x_{n_j}) < I(a_{n_j}) + \frac{1}{n_j}.$$

Since $\langle a_{n_j} \rangle$ can't cluster, neither can $\langle x_{n_j} \rangle$ as $\lim_{j\to\infty} d(a_{n_j}, x_{n_j}) = 0$. Thus, $C = \{a_{n_j} : j \in \mathbb{N}\}$ and $E = \{x_{n_j} : j \in \mathbb{N}\}$ are disjoint asymptotic closed subsets of X with $C \subseteq A$, once again violating condition (iii). $\square$

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