Baire Spaces and the Wijsman Topology

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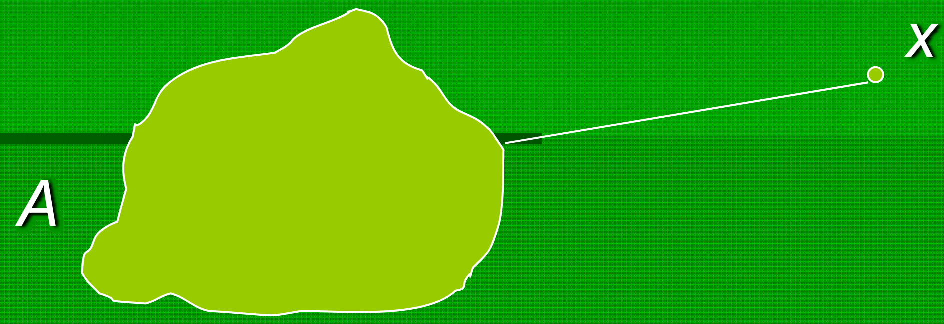
What is the Wijsman topology?

- It originates in the study of convergence of convex sets in Euclidean space by R. Wijsman in 1960s. The general setting was developed in 1980s.

- Let (X,d) be a metric space. If x is a point in X, and A is a nonempty closed subset of X:



The gap between A and x is: $d(A,x) = \inf \{d(a,x)\text{: } a \text{ in } A\}$.

- Define

  $F(X)$ = {all nonempty closed subset of X}.

  If A runs through $F(X)$ and x runs through X, we have a two-variable mapping: $d(\cdot,\cdot)$: $F(X) \times X \to R$.

- For any fixed A in $F(X)$, it is well-known (and simple) that $d(A,\cdot)$: $(X,d) \to (R, \mathcal{T}_u)$ is a continuous mapping.

- What will happen if we consider $d(\cdot, x)$: $F(X) \to (R, \mathcal{T}_u)$ for all fixed x in X ?

- The Wijsman topology $\mathcal{T}_{wd}$ on $F(X)$, is the weakest topology such that for each x in X, the mapping $d(\cdot,x)$ is continuous.

- By definition, a subbase for the Wijsman topology is

$$\Big\{\{A : d(A,x) < \epsilon\} : x \in X, \epsilon > 0\Big\} \bigcup \Big\{\{A : d(A,x) > \epsilon\} : x \in X, \epsilon > 0\Big\}.$$

- In the past 40 years, this topology has been extensively investigated. For more details, refer to the book, "*Topologies on closed and closed convex sets*", by G. Beer in 1993.

Some basic properties

- The Wijsman topology is not a topological invariant.

- For each A in $F(X)$, $A \leftrightarrow d(A,\cdot)$: $(X,d) \to (R, \mathcal{T}_u)$. Then, $(F(X), \mathcal{T}_{wd})$ embeds in $C_p(X)$ as a subspace. Thus, $(F(X), \mathcal{T}_{wd})$ is a Tychonoff space.

- Theorem For a metric space (X,d), the following are equivalent:

  (1) $(F(X), \mathcal{T}_{wd})$ is metrizable;

  (2) $(F(X), \mathcal{T}_{wd})$ is second countable;

  (3) $(F(X), \mathcal{T}_{wd})$ is first countable;

  (4) X is separable.

  (Cornet, Francaviglia, Lechicki, Levi)

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Completeness properties

Theorem (Effros, 1965): If X is a Polish space, then there is a compatible metric d on X such that $(F(X), \mathcal{T}_{wd})$ is Polish.

Theorem (Beer, 1991): If (X,d) is a complete and separable metric space, then $(F(X), \mathcal{T}_{wd})$ is Polish.

Theorem (Constantini, 1995): If X is a Polish space, then $(F(X), \mathcal{T}_{wd})$ is Polish for every compatible metric d on X.

Example (Constantini, 1998): There is a complete metric space (X,d) such that $(F(X), \mathcal{T}_{wd})$ is not Čech-complete.

Baire spaces

- A space X is Baire if the intersection of each sequence of dense open subsets of X is dense in X; and a Baire space X is called hereditarily Baire if each nonempty closed subspace of X is Baire.

  Baire Category Theorem: Every Čech-complete space is hereditarily Baire.

- Concerning Baireness of the Wijsman topology, we have the following known result:

  Theorem (Zsilinszky, 1998): If (X,d) is a complete metric space, then $(F(X), \mathcal{T}_{wd})$ is Baire.

Theorem (Zsilinszky, 20??): Let X be an almost locally separable metrizable space. Then X is Baire if and only if $(F(X), \mathcal{T}_{wd})$ is Baire for each compatible metric d on X.

- A space is almost locally separable, provided that the set of points of local separability is dense.

In a metrizable space, this is equivalent to having a countable-in-itself π-base.

Corollary (Zsilinszky, 20??): A space X is separable, metrizable and Baire if and only if $(F(X), \mathcal{T}_{wd})$ is a metrizable and Baire space for each compatible metric d on X.

- Motivated by the previous results, Zsilinszky posed two open questions. The first one is:

  Question 1 (Zsilinszky, 1998): Suppose that (X,d) is a complete metric space. Must $(F(X), T_{wd})$ be hereditarily Baire?

- Therefore, the answer to Question 1 is negative.

  Theorem (Chaber and Pol, 2002): Let X be a metrizable space such that the set of points in X without any compact neighbourhood has weight 2^{ω}. Then for any compatible metric d on X, $\mathbb{N}^{2^{\omega}}$ embeds in $(F(X), T_{wd})$ as closed subspace. In particular, the Wijsman hyperspace contains a closed copy of rationals.

- The second question was posed by Zsilinszky at the 10th Prague Toposym in 2006.

  Question 2 (Zsilinszky, 2006): Suppose that (X,d) is a metric hereditarily Baire space. Must $(F(X), \mathcal{T}_{wd})$ be Baire?

- The main purpose of this talk is to present an affirmative answer to Question 2.

  To achieve our goal, we shall need three tools:

- The ball proximal topology on $F(X)$.
- The pinched-cube topology on the countable power of X.
- A game-theoretic characterization of Baireness.

The ball proximal topology

- For each open subset E of X, we define

$$E^{++} = \{A \in CL(X) : d(A, X \setminus E) > 0\}.$$

The ball proximal topology $\mathcal{T}_{bpd}$ on $F(X)$ has a subbase

$$\{U^- : U \text{ is open }\} \cup \{(X \setminus B)^{++} : B \text{ is a proper closed ball}\},$$

where $U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}.$

- In general, $\mathcal{T}_{wd}$ is coarser than but different from $\mathcal{T}_{bpd}$ on F(X). Thus, the identity mapping ***i***: $(F(X), \mathcal{T}_{bpd}) \to (F(X), \mathcal{T}_{wd})$ is continuous, but is not open in general.

- For each nonempty open subset U in $(F(X), \mathcal{T}_{bpd})$, i(U) has nonempty interior in $(F(X), \mathcal{T}_{wd})$.

- A mapping $f: X \to Y$ is called feebly open if $f(U)$ has nonempty interior for every nonempty open set U of X.

- Similarly, a mapping $f: X \to Y$ is called feebly continuous whenever the pre-image of an open set is nonempty, the pre-image has a nonempty interior.

Theorem (Frolik, 1965; Neubrunn, 1977): Let $f: X \to Y$ be a feebly continuous and feebly open bijection. Then X is Baire if and only if Y is Baire.

Theorem (Zsilinszky, 20??): Let (X,d) be a metric space. Then $(F(X), \mathcal{T}_{bpd})$ is Baire if and only if $(F(X), \mathcal{T}_{wd})$ is Baire.

The pinched-cube topology

- The idea of pinched-cube topologies originates in McCoy's work in 1975.

- The term was first used by Piotrowski, Rosłanowski and Scott in 1983.

- Let (X,d) be a metric space. Define

  Δ_d = {finite unions of proper closed balls in X}.

- For B in Δ_d, and U_i ($i < n$) disjoint from B, we define

  $$[U_0, ..., U_{n-1}]_B = \prod_{i<n} U_i \times (X \setminus B)^{\omega \setminus n}.$$

  The collection of sets of this form is a base for a topology T_{pd} on X^ω, called the pinched-cube topology.

Theorem (Cao and Tomita): Let (X,d) be a metric space. If X^ω with $\mathcal{T}_{pd}$ is Baire, then $(F(X), \mathcal{T}_{bpd})$ is Baire.

- Let $\mathcal{S}(X)$ be the subspace of $(F(X), \mathcal{T}_{bpd})$ consisting of all separable subsets of X.

Define $f : X^\omega \to \mathcal{S}(X)$ by letting

$$f((x(k) : k < \omega)) = \overline{\{x(k) : K < \omega\}}.$$

It can be checked that f is continuous, and feebly open.

Then, it follows that $(F(X), \mathcal{T}_{bpd})$ is Baire.

The Choquet game

- Now, we consider the Choquet game played in *X*.

- There are two players, *β* and *α*, who select nonempty open sets in *X* alternatively with *β* starting first:

  $$U_0\ (\beta)\ \supseteq V_0\ (\alpha)\ \supseteq U_1\ (\beta)\ \supseteq V_1\ (\alpha)\ \cdots$$

  We claim that *α wins* a play if

  $$\bigcap_{n<\omega} U_n \left(= \bigcap_{n<\omega} V_n\right) \neq \emptyset.$$

  Otherwise, we say that *β has won* the play already.

- Without loss of generality, we can always restrict moves of *β* and *α* in a base or *π*-base of *X*.

- Baireness of a space can be characterized by applying the Choquet game.

Theorem (Oxtoby, Krom, Saint-Raymond): A space X is not a Baire space if and only if β has a winning strategy.

- This theorem was first given by J. Oxtoby in 1957. Then, it was re-discovered by M. R. Krom in 1974, and later on by J. Saint-Raymond in 1983.

- By a strategy for the player β, we mean a mapping defined for all finite sequences of moves of α. A winning strategy for β is a strategy which can be used by β to win each play no matter how α moves in the game.

Main results

Theorem (Cao and Tomita): Let (X,d) be a metric space. If X^ω with the Tychonoff topology is Baire, then X^ω with $\mathcal{T}_{pd}$ is also Baire.

- The basic idea is to apply the game characterization of Baireness.

Suppose that X^ω with the Tychonoff topology is Baire. Pick up any strategy σ for β in $\mathcal{T}_{pd}$.

We apply σ to construct inductively a strategy Θ for β in the Tychonoff topology. Since Θ is not a wining strategy for β in the Tychonoff topology, it can be show that σ is not a winning for β in $\mathcal{T}_{pd}$ either.

Theorem (Cao and Tomita): Let (X,d) be a metric and hereditarily Baire space. Then $(F(X), T_{wd})$ is Baire.

- Note that the Tychonoff product of any family $\{X_i : i \in I\}$ of metric hereditarily Baire spaces is Baire (Chaber and Pol, 2005).

It follows that X^ω with the Tychonoff topology is Baire, and thus X^ω with T_{pd} is Baire.

By one of the previous theorems, $(F(X), T_{pd})$ is a Baire space.

Finally, by another previous theorem, $(F(X), T_{wd})$ is Baire.

Comparison and examples

- For a space X, recall that the Vietoris topology $\mathcal{T}_V$ on $F(X)$ has

$$\{\langle U_0, \cdots U_{n-1}\rangle : U_0, \cdots, U_{n-1} \text{ are open sets of } X\}$$

as a base, where

$$\langle U_0, \cdots U_{n-1}\rangle = \{A \in CL(X) : A \subseteq \bigcup_{i<n} U_i, A \cap U_i \neq \emptyset\}.$$

- The relationship between the Vietoris topology and the Wijsman topologies is given in the next result.

Theorem (Beer, Lechicki, Levi, Naimpally, 1992): Let X be a metrizable space. Then, on $F(X)$,

$$\mathcal{T}_V = \sup\{\mathcal{T}_{Wd} : d \text{ is a compatibe metric on } X\}.$$

- For a metric space (X,d), we have the following diagram:

$$\begin{tikzcd}
 & X^{\omega} \text{ Baire} \arrow[dl] \arrow[dr] & \\
(F(X), \mathcal{T}_V) \text{ Baire} \arrow[dr] & & (F(X), \mathcal{T}_{wd}) \text{ Baire} \arrow[dl, "/" marking] \\
 & X^{n} \text{ Baire} &
\end{tikzcd}$$

Example (Pol, 20??): There is a separable metric (X,d) which is of the first Baire category such that $(F(X), \mathcal{T}_{wd})$ is Baire.

- Consider ω^ω equipped with the metric defined by

$$e(x,y) = \begin{cases} 0, & \text{if } x = y; \\ 2^{-n}, & n = \min\{m : x(m) \neq y(m)\} \text{ if } x \neq y. \end{cases}$$

Let $\omega^{<\omega}$ be the subspace consisting of sequences which are eventually equal to zero. This subspace is of the first Baire category.

The product space $X = \omega^{<\omega} \times \omega^\omega$ is a separable metric space which is of the first Baire category.

It can be shown that $(F(X), \mathcal{T}_{wd})$ is Baire.

Example (Cao and Tomita): There is a metric Baire space (X,d) such that $(F(X), T_{wd})$ is Baire, but X^ω with the Tychonoff topology is not Baire.

- Let $\{A_y : y \in \omega^\omega\}$ be a family of disjoint stationary subsets of $C_\omega \mathfrak{c}^+$. For each $y \in \omega^\omega$, we put $C_y = \bigcup\{A_{y'} : y' \in \omega^\omega,\ y(0) \neq y'(0)\}$. Let (ω^ω, e) be the metric space defined before. On $(\mathfrak{c}^+)^\omega$, we consider the metric defined by

$$\varrho(f,g) = \begin{cases} 0, & \text{if } f = g; \\ 2^{-(n+1)}, & n = \min\{m : f(m) \neq g(m)\} \text{ if } x \neq y. \end{cases}$$

Then, the required space will be

$$X = \{(y,f) \in \omega^\omega \times (\mathfrak{c}^+)^\omega : \sup\{f(n)+1 : n < \omega\} \in C_y\}$$

equipped with the product metric $e \times \varrho$ from $\omega^\omega \times (\mathfrak{c}^+)^\omega$.

Thank You