arXiv:submit/0915959 [math.PR] 19 Feb 2014

A debt behaviour model

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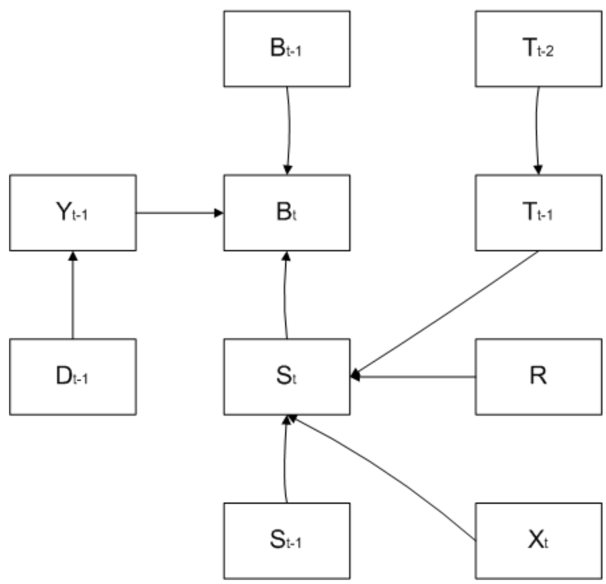

Figure 1: This diagram depicts the underlying causal structure of the model. See the text for the definitions of D,Y,B,T,S.

The model concerns the following random variables:

- A discrete Markov process B_t which records the *behavioural state* of the debtor during the time period t - measured in months. The state is measured in the middle of each month.
- A discrete-valued process T_t which records the *strongest debt management intervention* that was applied to the debtor during the time period t.

- R an entity-specific variable, R gives the final result of the debtor's most immediate previous debt case - NA, paid in full, liquidation/bankrupty, full write-off, partial write-off.

- X_t is the economic state at time period t. This measure is obtained through clustering a pertinent collection of economic variables: change in CPI, change in unemployment, change in the average weekly wage, etc. The underlying variables for X_t are varying quarterly, so X_t will be constant in blocks of three months.

- S_t is a latent discrete Markov process which categorizes debtors in a time period into the *behavioural scheme* that governs the generation of B_t. The model supposes that T_{t-1} influences S_t, and hence influences B_t indirectly.

- D_t is a positive real-valued variable, given by

$$D_t = \frac{\text{Debt amount at time } t\text{, including penalties and interest}}{\text{Largest amount of debt owed up to time } t\text{, excluding penalties and interest}}$$

- Y_t is a categorization of D_t into $\{0, 1\}$ - this is governed by a parameter α that needs to be inferred. the notion is that as a debtor gets closer to being paid in full, its probability of making a large lump-sum payment to clear its debt may change.

We introduce a set of parameters as follows:

- α: defined by $Y_t := 0$ if and only if $D_t \leq \alpha$.

- Q_S: a list of transition matrices, one for each combination of values of R, X_t, T_{t-1}.

- π_S: a list of initial probabilities, one for each combination of values of R, X_t.

- Q_B: a list of transition matrices, one for each combination of values of Y_{t-1} and S_t.

- π_B: a list of initial probabilities, one for each value of S_1.

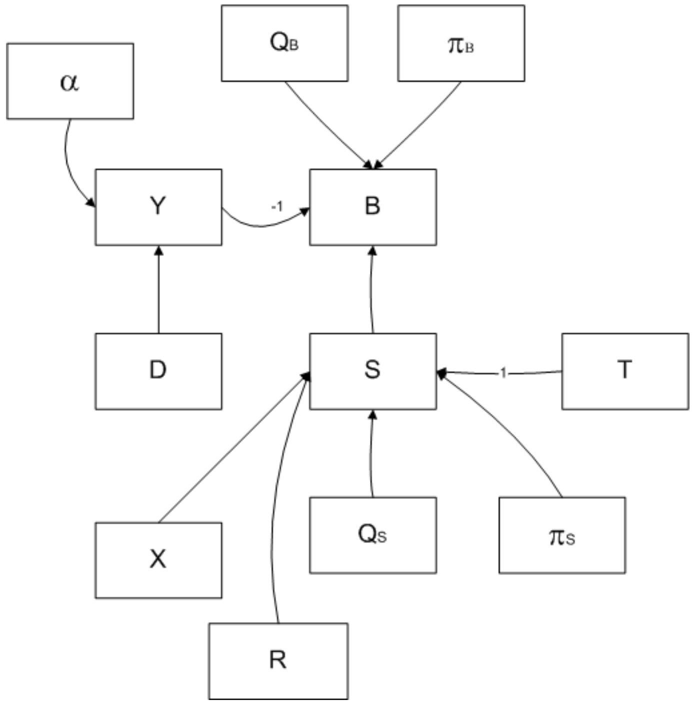

Figure 2: This diagram depicts the underlying causal structure of the model, including the parameters. Refer to the text for definitions of the parameters $\pi_B, Q_B, \pi_S, Q_S, \alpha$

Figure 2 depicts the causal structure of the variables and the parameters - we have now expressed each of the variables as a vector of length as long as the number of observation periods.

Every debt case begins at a time period u and ends at a time period l. If the debt case is indexed by i, the the beginning is u_i and the end is l_i. There will be observations of T_t, B_t, D_t, and X_t from u_i through to l_i.

The log-likelihood of observing a single debt case is maximized when we maximize:

$$l_0 = \sum_{t=u+1}^{t=l} (\ln(Q_B^{Y_{t-1},S_t}(B_{t-1}, B_t)) + \ln(Q_S^{X_t,R,T_{t-1}}(S_{t-1}, S_t))) + \ln(\pi_B^{S_u}(B_u)) + \ln(\pi_S^{X_u,R}(S_u))$$

We apply the EM algorithm to l_0, taking the expected value of l_0 conditional on $\{B_t, X_t, D_t, T_t, R\}$ and the k-th iteration of the parameters $\{\alpha, Q_B, Q_S, \pi_B, \pi_S\}$,

Θ^k.

For this we define the *responsibilities* for each debt case, i, and time t, $t = u_i, \ldots, l_i$:

$$\gamma_{i,t}(s) := p(S_t = s | T_{u_i}^{l_i - 1}, X_{u_i}^{l_i}, B_{u_i}^{l_i}, R_i, D_{u_i}^{l_i - 1})$$

for $t \geq u_i$; and for $t > u_i$,

$$\Gamma_{i,t}(p,q) := p(S_t = q, S_{t-1} = p | T_{u_i}^{l_i - 1}, B_{u_i}^{l_i}, R_i, D_{u_i}^{l_i - 1})$$

It is clear that $\gamma_{i,t}(s) = \sum_p \Gamma_{i,t}(p,s)$, or if $t = u_i$, $\gamma_{i,u_i}(s) = \sum_q \Gamma_{i,u_i+1}(s,q)$ - hence we need only compute $\Gamma_{i,t}$.

This is done using the Forward-Backward algorithm:

1 Calculating $\Gamma_{i,t}$

This calculation is standard, but we present it for completeness.

Define the following four sets of probabilities:

- $\pi_t(s) = p(S_t = s | T_u^{l-1}, X_u^l, R, D_u^{l-1}, B_u^l)$
- $\pi'_t(s) = p(S_t = s | T_u^{t-1}, X_u^t, R, D_u^{t-1}, B_u^t)$, $t \geq u$.
- $F_t(p,q) = p(S_{t-1} = p, S_t = q | T_u^{t-1}, X_u^t, R, D_u^{t-1}, B_u^t)$, $t > u$
- $\Gamma_t(p,q) = p(S_{t-1} = p, S_t = q | T_u^{l-1}, X_u^l, R, D_u^{l-1}, B_u^l)$,$t > u$.

Then

$$\begin{aligned} F_t(p,q) &\propto Q_B^{q,Y_{t-1}}(B_{t-1}, B_t) Q_S^{T_{t-1}, X_t, R}(p,q) \pi'_{t-1} \\ &= (Q_B^{q,0}(B_{t-1}, B_t) I_{[0,\alpha]}(D_{t-1}) + Q_B^{q,1}(B_{t-1}, B_t) I_{(\alpha,\infty)}(D_{t-1})) Q_S^{T_{t-1}, X_t, R}(p,q) \end{aligned}$$

and

$$\pi'_t(q) = \sum_p F_t(p,q)$$

with $\pi'_u(s) \propto \pi_B^s(B_u) \pi_S^{X_u, R}(s)$. The normalizing constants can be found by noting that $\sum_{p,q} F_t(p,q) = 1$ and $\sum_s \pi'_u(s) = 1$.

Having obtained $F_t(p,q)$ (the *forward matrices*) we can calculate the *backward matrices* Γ_t as follows:

Set $\Gamma_l = F_l$.
For $t < l$,

$$
\begin{aligned}
\Gamma_t(p,q) &= p(S_{t-1}=p|S_t=q, T_u^{l-1}, X_u^l, R, D_u^{l-1}, B_u^l) p(S_t=q|T_u^{l-1}, X_u^l, R, D_u^{l-1}, B_u^l) \\
&= p(S_{t-1}=p|S_t=q, T_u^{t-1}, X_u^t, R, D_u^{t-1}, B_u^t)\pi_t(q) \\
&= F_t(p,q)\frac{\pi_t(q)}{\pi'_t(q)}
\end{aligned}
$$

2 Update equations for the M-step

The formulas that follow are the result of straightforward calculations.

$$
\begin{aligned}
Q_B^{s,y}(b,c) &= \frac{\sum_i \sum_{t=u_i+1}^{l_i} \delta(B_{i,t}-c)\delta(B_{i,t-1}-b)\delta(Y_{i,t-1}-y)\gamma_{i,t}(s)}{\sum_i \sum_{t=u_i+1}^{l_i} \delta(B_{i,t-1}-b)\delta(Y_{i,t-1}-y)\gamma_{i,t}(s)} \\
\pi_B^s(b) &= \frac{\sum_i \delta(B_{i,u_i}-b)\gamma_{i,u_i}(s)}{\sum_i \gamma_{i,u_i}(s)} \\
Q_S^{T,R,X}(p,q) &= \frac{\sum_i \sum_{t=u_i}^{l_i-1} \delta(T_{i,t}-T)\delta(R_i-R)\delta(X_t-X)\gamma_{i,t}(p)\gamma_{i,t+1}(q)}{\sum_i \sum_{t=u_i}^{l_i-1} \delta(T_{i,t}-T)\delta(X_t-X)\delta(R_i-R)\gamma_{i,t}(p)} \\
\pi_S^{R,X}(s) &= \frac{\sum_i \delta(R_i-R)\delta(X_{u_i}-X)\gamma_{i,u_i}(s)}{\sum_i \delta(R_i-R)\delta(X_{u_i}-X)}
\end{aligned}
$$

Note that Q_B depends on an unknown value of α. The approach will be to fit Q_B for a range of values of α, and to choose the α that gives the maximum value to:

$$
l_1 = \sum_i \sum_{t=u_i+1}^{l_i} \sum_s \ln(Q_B^{s,0}(B_{i,t-1}, B_{i,t}) I_{[0,\alpha]}(D_{i,t-1}) + Q_B^{s,1}(B_{i,t-1}, B_{i,t}) I_{(\alpha,\infty)}(D_{i,t-1}))\gamma_{i,t}(s)
$$