Weak Continuity Properties of Topologized Groups

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This is joint work with R. Drozdowski and Z. Piotrowski

Topologized groups

A topologized group is a triple $(G, \cdot, \tau)$ such that $(G, \cdot)$ is a group and (G, τ) is a topological space. If both the multiplication $\mathfrak{m}$ and the inversion $\mathfrak{i}$ of G are continuous, then $(G, \cdot, \tau)$ is called a topological group.

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If $\mathfrak{m}$ is continuous, then $(G, \cdot, \tau)$ is called a paratopological group. In case that $\mathfrak{m}$ is separately continuous, then $(G, \cdot, \tau)$ is called a semitopological group. If left translations are continuous, $(G, \cdot, \tau)$ is called a left semitopological group. Right semitopological groups can be defined similarly.

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It is known that weaker and less restrictive assumptions which can be used to characterize a group topology.

Montgomery's theorem

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This suggests the question of studying “nice" topological conditions on $(G, \cdot, \tau)$ to make $(G, \cdot, \tau)$ a topological group. Also, the question as to when a separately continuous mapping is (jointly) continuous arises.

Arhangel'skii and Reznichenko

In 2005, Arhangel'skii and Reznichenko obtained the following result.

Arhangel'skii-Reznichenko Theorem: Every Hausdorff paratopological group $(G, \cdot, \tau)$ such that (G, τ) is a symmetrizable Baire space is a metrizable topological group.

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To show this theorem, they employed the condition:

$$e \in \overline{\mathrm{int}(\overline{U^{-1}})}$$

for every open neighborhood U of the neutral element e.

Ferri, Hernández and Wu

In 2006, Ferri, Hernández and Wu considered topologized groups with a Baire metrizble topology, and use some weaker conditions on left and right translations to characterize a group topology.

Almost continuity: A mapping $f : X \to Y$ is called almost continuous if for each non-empty open subset V of Y, $f^{-1}(V)$ contains a non-empty open subset of X.

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First, this definition has a little flaw. Second, we know that this concept has appeared in the literature under two different names: feebly continuous (Frolík, 1961) and somewhat continuous (Gentry and Hoyle, 1971).

Near and quasi- continuity

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Nearly continuity: A mapping $f : X \to Y$ is called nearly continuous at $x \in X$ if for every open neighborhood V of $f(x)$, then set $\overline{f^{-1}(V)}$ is a neighborhood of x (Pták, 1958).

Quasi-continuity: A mapping $f : X \to Y$ is called quasi-continuous at $x \in X$ if for every open neighborhood V of $f(x)$ and every open neighborhood U of x, there is an open set $O \subseteq U$ such that set $O \subseteq f^{-1}(V)$ (Kempsity, 1932).

Bouziad and Reznichenko

In 1990s, the nearly and quasi- continuity came to the play, due to Bouziad and Reznichenko.

Bouziad-Reznichenko Theorem: For a paratopological group $(G, \cdot, \tau)$, the following are equivalent:

(i) $(G, \cdot, \tau)$ a topological group;

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The reference for the equivalence of (i) and (ii) is a paper by Kenderov, Korezov and Moors in 2001.

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To answer these questions, we need the following concept introduced by Andrijević in 1986.

Semi-precontinuity: A subset A of X is semi-preopen if $A \subseteq \overline{\mathrm{int}(\overline{A})}$.

A mapping $f : X \to Y$ is called semi-precontinuous at $x \in X$ if for every open neighborhood V of $f(x)$, $f^{-1}(V)$ is a semi-preopen set in X.

The first theorem

Theorem 1. Let $(G, \cdot, \tau)$ be a paratopological group. Then the following statements are equivalent.

(i) $(G, \cdot, \tau)$ is a topological group;

(ii) $\mathfrak{i}$ is quasi-continuous;

(iii) $\mathfrak{i}$ of $(G, \cdot, \tau)$ is nearly continuous;

(iv) $\mathfrak{i}$ is semi-precontinuous;

(v) For every open neighbourhood U of e, $e \in \overline{\operatorname{int}\left(\overline{U^{-1}}\right)}$;

(vi) For every open neighbourhood U of e, $e \in \operatorname{int}\overline{(U \cap U^{-1})}$;

(vii) For every open neighbourhood U of e, $\operatorname{int}\overline{(U \cap U^{-1})} \neq \varnothing$.

Comments and remarks

There are paratopological groups with a feebly continuous inversion, which are not topological groups. In deed, Guran called such paratopological groups saturated. In a series of papers, Banakh and Ravsky showed that saturated paratopological groups behave much like topological groups in many aspects.

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We may consider the “dual problem" of Theorem 1. That is, for a given topologized group $(G, \cdot, \tau)$ with a continuous inversion, what can we say about the weak continuity properties of the multiplication $\mathfrak{m}$?

We will see that, to study weak continuity properties of $\mathfrak{m}$, we need put more conditions on the topology τ.

Two examples

Example 1. Let $(\mathbb{Z}_2, +)$ be the group of integers modulo 2 with the usual addition. Let τ be the Sierpiński topology on $\mathbb{Z}_2$. Then $(\mathbb{Z}_2, +, \tau)$ is a topologized group such that $\mathfrak{m}$ is both quasi-continuous and nearly continuous. Also, $\mathfrak{i}$ is continuous. But, $(\mathbb{Z}_2, +, \tau)$ is not a paratopological group.

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Example 2. Let $(\mathbb{R}, +)$ be the reals equipped with the usual addition and the following metric

$$d(x,y) = \begin{cases} 0, & \text{if } x = y; \\ \max\{|x|, |y|\}, & x \neq y. \end{cases}$$

Then $(\mathbb{R}, +, \tau_d)$ is a topologized group such that $\mathfrak{m}$ is feebly continuous, but not quasi-continuous. Moreover, it can be checked easily that $\mathfrak{i}$ is continuous.

The Novak number

The Novak number of a space X is defined by

$$\mathrm{Nov}(X) = \min\{\kappa : X \text{ is covered by } \leq \kappa \text{ n. d. sets}\}.$$

Then X is of second Baire category iff $\mathrm{Nov}(X) \geq \aleph_1$, and X is Baire if and only if $\mathrm{Nov}(O) \geq \aleph_1$ for any nonempty open set $O \subseteq X$.

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A classical result says that a homogeneous space is Baire if and only if it is of second category. Similarly, one can show the following:

If $(G, \cdot, \tau)$ is a left (or right) semitopological group, then $\mathrm{Nov}(G) = \mathrm{Nov}(O)$ for every nonempty open set $O \subseteq G$.

Developability number

The developability number of a space X, is defined by

$$\begin{aligned}\operatorname{dev}(X) &= \min\{\kappa : \exists \text{ open covers } \{\mathscr{U}_\alpha : \alpha < \kappa\} \text{ of } X \\ &\qquad \text{s. t. } \{\operatorname{st}(x, \mathscr{U}_\alpha) : \alpha < \kappa\} \text{ is a local base at } x\}.\end{aligned}$$

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Corollary 2.1. If $(G, \cdot, \tau)$ is a left (or right) semitopological group such that (G, τ) is a Baire and Moore space, then $\mathfrak{i}$ is nearly continuous.

The third theorem

Theorem 3. Let $(G, \cdot, \tau)$ be a topologized group endowed with a regular topology τ such that $\mathrm{dev}(G) < \mathrm{Nov}(G)$. If one type of translations are feebly continuous, and the other type of translations are continuous, then $(G, \cdot, \tau)$ is a topological group.

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As an immediate consequence, we have the following corollary:

Corollary 3.1. Let $(G, \cdot, \tau)$ be a topologized group such that (G, τ) is a Baire Moore space. If one type of translations are feebly continuous, and the other type of translations are continuous, then $(G, \cdot, \tau)$ is a metrizable topological group.

Comparison

It is easy to see that our results extend the following two theorem:

Piotrowski's Theorem: Every Baire Moore semitopological group is a paratopological group (1998).

Reznichenko's Theorem: Every semitopological group which is a Baire metrizable space is a topological group.

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Piotrowski's Theorem: Every Baire Moore semitopological group is a paratopological group (1998).

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Comparing our results with the Arhangel'skii-Reznichenko theorem, we have the following question.

Question 1. Must every symmetrizable Hausdorff Baire semitopological group be a topological group?

Ferri-Hernández-Wu Theorem

Ferri-Hernández-Wu Theorem: Let G be a group, which is a Baire metrizable space. Suppose that there is a dense subset S of second category in G such that the right translations ρ_s and $\rho_{s^{-1}}$ are continuous for all $s \in S$. Suppose further that, for each $s \in G$, there is a residual subset R_s of G such that the left translation λ_s is feebly continuous on R_s, $\lambda_s(R_s)$ is residual, and $\lambda_{s^{-1}}$ is feebly continuous on $\lambda_s(R_s)$. Then G is a topological group.

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Question 2. Can we relax the condition all right translations are continuous to there is a dense subset S of second category in G such that the right translations ρ_s and $\rho_{s^{-1}}$ are continuous for all $s \in S$?

One more example

In general, the conclusion of Theorem 3 may not hold for a topologized group $(G, \cdot, \tau)$ when both left and right translations are feebly continuous, under the same assumption on the topology τ.

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Example 3. Let $G = [0, 1)$ be equipped with the following multiplication operation

$$x \cdot y = \begin{cases} x + y, & \text{if } x + y < 1; \\ x + y - 1 & x + y \geq 1 \end{cases}$$

and the usual topology τ. Then, $(G, \cdot, \tau)$ is a topologized group with a separable metrizable Baire topology. It can be checked that $\mathfrak{m}$ is separately quasi-continuous, but neither $\mathfrak{m}$ not $\mathfrak{i}$ is continuous.

Two more questions

In Example 3, it can be checked that $\mathfrak{m}$ is not continuous at any point of $\{(x, y) \in G \times G : x + y = 1\}$. However, this outcome is somehow not surprising.

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Question 3. Let $(G, \cdot, \tau)$ be a topologized group with a Polish topology τ such that $\mathfrak{m}$ is separately quasi-continuous. Must $(G, \cdot, \tau)$ be a topological group?

Here is another question that we could not solve at the moment.

Question 4. Let $(G, \cdot, \tau)$ be a topologized group equipped with a separable, metrizable and Baire topology τ such that $\mathfrak{m}$ is separately feebly continuous. Must $\mathfrak{m}$ be feebly continuous?

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Question 4. Let $(G, \cdot, \tau)$ be a topologized group equipped with a separable, metrizable and Baire topology τ such that $\mathfrak{m}$ is separately feebly continuous. Must $\mathfrak{m}$ be feebly continuous?

If the answer is “yes", then $C(\mathfrak{m})$ is somewhere dense, and thus non-empty. Therefore, we can view Question 4 as an analog of Talagrand’s problem in 1985.

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Thank you very much!