MARKOV CHAIN PROPERTIES IN TERMS OF COLUMN SUMS OF THE TRANSITION MATRIX

JEFFREY J HUNTER

SCHOOL OF COMPUTING & MATHEMATICAL SCIENCES

AUT UNIVERSITY
TE WĀNANGA ARONUI O TAMAKI MAKAU RAU

WWPMS 2011

Abstract

Questions are posed regarding the influence that the column sums of the transition probabilities of a stochastic matrix (with row sums all one) have on the stationary distribution, the mean first passage times and the Kemeny constant of the associated irreducible discrete time Markov chain. Some new relationships, including some inequalities, and partial answers to the questions, are given using a special generalized matrix inverse that has not previously been considered in the literature on Markov chains.

Outline

1. Introduction
2. Properties of the generalized inverse, *H*
3. Stationary distributions
4 Relationship between *H and Z*
5. Mean first passage times
6. Kemeny's constant
7. Examples

Introduction

Let $\{X_n\}$, $(n \geq 0)$ be a finite irreducible, discrete time MC.

Let $S = \{1, 2, \ldots, m\}$ be its state space.

Let $P = [p_{ij}]$ be the transition matrix of the MC.

$\{X_n\}$ has a unique stationary distribution $\{\pi_j\}, j \in S$.

and finite mean first passage times $\{m_{ij}\}$, $(i, j) \in S \times S$.

P stochastic $\Rightarrow$ the row sums are all one.

$$\Rightarrow \sum_{j=1}^{m} p_{ij} = 1, \;\; i \in S.$$

Let $\{c_j\}$ be the column sums of the transition matrix.

$$\Rightarrow c_j = \sum_{i=1}^{m} p_{ij}, \; j \in S.$$

Questions

What influence does the sequence $\{c_j\}$ have on $\{\pi_j\}$?

What influence does the sequence $\{c_j\}$ have on $\{m_{ij}\}$?

Are there relationships connecting the $\{c_j\},\{\pi_j\},\{m_{ij}\}$?

Can we deduce bounds on the $\{\pi_j\}$ and the $\{m_{ij}\}$ involving the $\{c_j\}$?

What effect does the $\{c_j\}$ have on Kemeny's constant

$$K = \sum_{j=1}^{m} \pi_j m_{ij}\ ?$$

Technique

We use the generalized matrix inverse

$$H \equiv [\,I - P + \boldsymbol{e}\boldsymbol{c}^T\,]^{-1}$$

where $\boldsymbol{e}$ is the column vector of ones and

$\boldsymbol{c}^T = (c_1, c_2, \ldots, c_m)$ is the row vector of column sums of P.

Let $\Pi = \boldsymbol{e}\boldsymbol{\pi}^T$ where $\boldsymbol{\pi}^T = (\pi_1, \pi_2, \ldots, \pi_m)$.

H can also be expressed in terms of Kemeny and Snell's fundamental matrix

$$Z = [I - P + \Pi]^{-1},$$

or in terms of Meyer's group g-inverse

$$(I - P)^{\#} = Z - \Pi.$$

Previous results

1. Kirkland considers the subdominant eigenvalue λ_2 associated with the set S(c) of $m \times m$ stochastic matrices with column sum vector $\boldsymbol{c}^T$. The quantity $\overline{\lambda(c)} = \max\{|\lambda_2(A)| \,|\, A \in S(c)\}$ is considered. The vectors $\boldsymbol{c}^T$ such that $\overline{\lambda(c)} < 1$ are identified. In those cases, nontrivial upper bounds on $\overline{\lambda(c)}$ and weak ergodocity results for forward products are provided.

Previous results

2. Kirkland considers an irreducible stochastic matrix P and studies the extent to which the column sum vector for P provides information on a certain condition number $\kappa(P)$, which measures the sensitivity of the stationary distribution vector to perturbations in P.

Properties of H

$\pi^T \boldsymbol{t} \neq 0$ and $\boldsymbol{u}^T \boldsymbol{e} \neq 0 \Leftrightarrow I - P + \boldsymbol{t}\boldsymbol{u}^T$ is non-singular.

$[I - P + \boldsymbol{t}\boldsymbol{u}^T]^{-1}$ is a generalised inverse of $I - P$.

$\pi^T \boldsymbol{e} = 1 \neq 0$ and $\boldsymbol{c}^T \boldsymbol{e} = m \neq 0 \Rightarrow I - P + \boldsymbol{e}\boldsymbol{c}^T$ is non-singular.

$H = [I - P + \boldsymbol{e}\boldsymbol{c}^T]^{-1}$ is a generalized inverse of $I - P$.

Key properties of H

If $H = [I - P + \mathbf{e}\mathbf{c}^T]^{-1} = [h_{ij}]$ then $\mathbf{c}^T H = \boldsymbol{\pi}^T$.

Thus $\pi_j = \sum_{i=1}^{m} c_i h_{ij}$ for all $j \in S$.

Further $H\mathbf{e} = \mathbf{e}/m$ so that

$h_{i\bullet} \equiv \sum_{j=1}^{m} h_{ij} = 1/m$ for all $i \in S$.

Note also that $\mathbf{c}^T H\mathbf{e} = 1$.

Proof of key properties of H

$$(I - P + \mathbf{e}\mathbf{c}^T)H = I \Rightarrow H - PH + \mathbf{e}\mathbf{c}^T H = I.$$

Premultiply by $\boldsymbol{\pi}^T$. Since $\boldsymbol{\pi}^T P = \boldsymbol{\pi}^T$,

$$\Rightarrow \mathbf{c}^T H = \boldsymbol{\pi}^T \Rightarrow \pi_j = \sum_{i=1}^{m} c_i h_{ij} \text{ for all } j \in S.$$

$$H(I - P + \mathbf{e}\mathbf{c}^T) = I \Rightarrow H - HP + H\mathbf{e}\mathbf{c}^T = I.$$

Postmultiply by $\mathbf{e}$. Since $P\mathbf{e} = \mathbf{e}$ and $\mathbf{c}^T\mathbf{e} = m$,

$$\Rightarrow H\mathbf{e} = \mathbf{e}/m \Rightarrow h_{i\bullet} \equiv \sum_{j=1}^{m} h_{ij} = 1/m, \text{ for all } i \in S.$$

Properties of the elements of H

Let $\mathbf{e}_i^T$ ($\mathbf{e}_j$) be the i-th (j-th) elementary row (col) vector.

Let $\boldsymbol{h}_j^{(c)} \equiv H\mathbf{e}_j$ denote the j-th column of H.

Let $h_{\bullet j} = \mathbf{e}^T \boldsymbol{h}_j^{(c)}$ be the sum of the elements of the j-th col.

Let $\boldsymbol{h}_i^{(r)T} \equiv \mathbf{e}_i^T H$ denote the i-th row of H.

Let $h_{i\bullet} = \boldsymbol{h}_i^{(r)T}\mathbf{e}$ be the sum of the elements of the i-th row.

Let $\boldsymbol{h}_{rowsum} = H\mathbf{e} = \sum_{j=1}^{m} \boldsymbol{h}_j^{(c)} = [h_{1\bullet}, h_{2\bullet}, \ldots, h_{m\bullet}]^T$,

Let $\boldsymbol{h}_{colsum}^T = \mathbf{e}^T H = \sum_{j=1}^{m} \boldsymbol{h}_j^{(r)T} = [h_{\bullet 1}, h_{\bullet 2}, \ldots, h_{\bullet m}]$,

$h_{ij} = \mathbf{e}_i^T H \mathbf{e}_j$.

Properties of the elements of *H*

(a) (Row properties)

$$\boldsymbol{h}_i^{(r)T} - \boldsymbol{p}_i^{(r)T} H = \mathbf{e}_i^T - \boldsymbol{\pi}^T,$$

$$\boldsymbol{h}_i^{(r)T} - \boldsymbol{h}_i^{(r)T} P = \mathbf{e}_i^T - \boldsymbol{c}^T / m,$$

and $h_{i\bullet} = 1/m$.

(b) (Column properties)

$$\boldsymbol{h}_j^{(c)} - P\boldsymbol{h}_j^{(c)} = \mathbf{e}_j - \pi_j \mathbf{e},$$

$$\boldsymbol{h}_j^{(c)} - H\boldsymbol{p}_j^{(c)} = \mathbf{e}_j - (c_j / m)\mathbf{e},$$

and $h_{\bullet j} = 1 - (m-1)\pi_j$.

Properties of the elements of *H*

(c) (Element properties)

$$h_{ij} = \sum_{k=1}^{m} p_{ik} h_{kj} + \delta_{ij} - \pi_j,$$

$$h_{ij} = \sum_{k=1}^{m} h_{ik} p_{kj} + \delta_{ij} - c_j / m.$$

(d) (Row and Column sum properties)

$$\boldsymbol{h}_{rowsum} = \mathbf{e}/m,$$

$$\boldsymbol{h}_{colsum}^{T} = \mathbf{e}^{T} - (m-1)\boldsymbol{\pi}^{T}.$$

Explicit row and column sums of the elements of *H*.

Explicit expressions for individual h_{ij} not readily available.

Stationary distributions

MC irreducible

$\Longrightarrow$ exists a unique stationary distribution $\{\pi_j\}, j \in S$.

The stationary probabilities found as the solution of the stationary equations:

$$\pi_j = \sum_{i=1}^{m} \pi_i p_{ij} \quad (j \in S) \text{ with } \sum_{i=1}^{m} \pi_i = 1.$$

We have shown

$$\pi_j = \sum_{i=1}^{m} c_i h_{ij} \text{ with } \sum_{i=1}^{m} c_i = m.$$

Doubly stochastic matrices

Let the stationary probability vector be

$\boldsymbol{\pi}^T = (\pi_1, \pi_2, \ldots, \pi_m)$ so that $\boldsymbol{\pi}^T = \boldsymbol{\pi}^T P$ with $\boldsymbol{\pi}^T \mathbf{e} = 1$.

For doubly stochastic P, $\mathbf{e}^T = \mathbf{e}^T P$ so that $\boldsymbol{c}^T = \mathbf{e}^T$,

$$\boldsymbol{c} = \mathbf{e} \Leftrightarrow \boldsymbol{\pi} = \mathbf{e}/m.$$

$c_i = 1$ for all $i \in S \Leftrightarrow \pi_i = 1/m$ for all $i \in S$.

Generalized inverses

Kemeny and Snell's fundamental matrix of ergodic MCs

$Z = [I - P + \Pi]^{-1}$.

Meyer's group inverse of $I - P$,

$(I - P)^{\#} = Z - \Pi$.

Z and $(I - P)^{\#}$ are generalized inverses of $I - P$.

If G is any generalized inverse of $I - P$,

$(I - P)G(I - P)$ is invariant and $= (I - P)^{\#}$.

Relationship between *H* and *Z*

If $H = [I - P + \mathbf{e}\mathbf{c}^T]^{-1}$ and $Z \equiv [I - P + \mathbf{e}\boldsymbol{\pi}^T]^{-1}$ then

(a) $Z = H + \Pi - \Pi H,$

(b) $H = Z + \dfrac{1}{m}\Pi - \dfrac{1}{m}\mathbf{e}\mathbf{c}^T Z,$

(c) $(1 + m)\Pi = m\Pi H + \mathbf{e}\mathbf{c}^T Z,$

(d) $(1 + m)\boldsymbol{\pi}^T = m\boldsymbol{\pi}^T H + \mathbf{c}^T Z.$

Elemental relationships

If $H = [h_{ij}] = [I - P + \mathbf{e}\mathbf{c}^T]^{-1}$

and $Z = [z_{ij}] = [I - P + \mathbf{e}\boldsymbol{\pi}^T]^{-1}$,

then

$$(a) \quad z_{ij} = h_{ij} + \pi_j - \sum_{k=1}^{m} \pi_k h_{kj},$$

$$(b) \quad h_{ij} = z_{ij} + \frac{1}{m}\pi_j - \frac{1}{m}\sum_{k=1}^{m} c_k z_{kj},$$

$$(c) \quad (1+m)\pi_j = m\sum_{k=1}^{m} \pi_k h_{kj} + \sum_{k=1}^{m} c_k z_{kj}.$$

Properties of Z and H

For ergodic Markov chains the diagonal elements of Z, z_{ij}, are positive.

Matlab examples show that a similar relationship holds for the diagonal elements of H, h_{ij}.

(Formally established later.)

$$z_{ij} - h_{ij} = \pi_j - \sum_{k=1}^{m} \pi_k h_{kj} = \left(\sum_{k=1}^{m} c_k z_{kj} - \pi_j\right) \Big/ m,$$

i.e. $z_{ij} - h_{ij}$ is independent of i, and thus $= z_{jj} - h_{jj}$.

Consequently, $z_{jj} - z_{ij} = h_{jj} - h_{ij}$ for all i, j.

Mean first passage times

For an irreducible finite MC with transition matrix P,

let $M = \left[m_{ij} \right]$ be the matrix of expected first passage

times from state i to state j.

M satisfies the matrix equation

$$(I - P)M = E - PM_d,$$

where $E = \mathbf{ee}^T = [1]$, $M_d = [\delta_{ij} m_{ij}] = (\Pi_d)^{-1} \equiv$ D.

If G is any g-inverse of $I - P$, then

$$M = [G\Pi - E(G\Pi)_d + I - G + EG_d]D.$$

Mean first passage times

Under any of the following three equivalent conditions:

(i) $G\mathbf{e} = g\mathbf{e}$, g a constant,

(ii) $GE - E(G\Pi)_d D = 0$,

(iii) $G\Pi - E(G\Pi)_d = 0$,

$$M = [I - G + EG_d]D.$$

H satisfies (i) ($H\mathbf{e} = (1/m)\mathbf{e}$ and $g = 1/m$)

$$\Rightarrow M = [I - H + EH_d]D.$$

Z satisfies (i) (since $Z\mathbf{e} = \mathbf{e}$ and $g = 1$)

$$\Rightarrow M = [I - Z + EZ_d]D.$$

Mean first passage times

Thus

$$m_{ij} = \begin{cases} \dfrac{1}{\pi_j} = \dfrac{1}{\sum_{i=1}^{m} c_i h_{ij}}, & i = j, \\ \dfrac{h_{jj} - h_{ij}}{\pi_j} = \dfrac{h_{jj} - h_{ij}}{\sum_{i=1}^{m} c_i h_{ij}}, & i \neq j. \end{cases}$$

Thus a knowledge of the $\{c_i\}$ and the $\{h_{ij}\}$ leads directly to expressions for the $\{m_{ij}\}$.

New relationships

For all $j \in \{1,2,\ldots,m\}$,

$$\sum_{i=1}^{m} m_{ij} - \sum_{i \neq j} c_i m_{ij} = m.$$

– a new connection between the $\{c_j\}$ and $\{m_{ij}\}$.

$$\sum_{i=1}^{m} c_i m_{ij} = \frac{c_j}{\pi_j} - 1 + \frac{m h_{jj}}{\pi_j},$$

$$\sum_{i=1}^{m} m_{ij} = m_{\bullet j} = m - 1 + \frac{m h_{jj}}{\pi_j}.$$

Expressions for $\{\pi_j\}$

For all $j \in \{1,2,\ldots,m\}$,

$$\pi_j = \frac{c_j}{m - \sum_{i=1}^{m} m_{ij} + \sum_{i=1}^{m} c_i m_{ij}} = \frac{1}{m - \sum_{i \neq j} m_{ij} + \sum_{i \neq j} c_i m_{ij}}.$$

$$\pi_j = \frac{c_j + m h_{jj}}{1 + \sum_{i=1}^{m} c_i m_{ij}} = \frac{m h_{jj}}{1 + \sum_{i \neq j} c_i m_{ij}},$$

$$\pi_j = \frac{m h_{jj}}{1 + \sum_{i=1}^{m} m_{ij} - m} = \frac{m h_{jj} - 1}{1 + \sum_{i \neq j} m_{ij} - m}.$$

Positivity of the z_{jj} and h_{jj}

$$\boldsymbol{\pi}^T M = \mathbf{e}^T Z_d D = \left(z_{11}/\pi_1, \ldots, z_{jj}/\pi_j, \ldots, z_{mm}/\pi_m \right)$$

$\Rightarrow z_{jj} > 0$ for all j.

$$m_{jj} m h_{jj} = 1 + \sum_{i \neq j}^{m} c_i m_{ij} > 0 \Rightarrow h_{jj} > 0 \text{ for all } j.$$

Since $\pi_j m_{ij} = h_{jj} - h_{ij} = z_{jj} - z_{ij} > 0 \Rightarrow h_{jj} > h_{ij}$.

No surety regarding the sign of any of the $\{h_{ij}\}$ for $i \neq j$.

Doubly stochastic matrices

If $c_i = 1$ for all i, then

$$m_{\bullet j} = m - 1 + m^2 h_{jj} = m^2 z_{jj}.$$

Also

$$m_{\bullet j} \leq m_{\bullet i} \Leftrightarrow z_{jj} \leq z_{ii} \Leftrightarrow h_{jj} \leq h_{ii}.$$

Kemeny's constant

The expression $K_i = \sum_{j=1}^{m} m_{ij}\pi_j$ is in fact independent of $i \in \{1,2,....,m\} \Rightarrow K_i = K$, "Kemeny's constant".

K has many important interpretations in terms of properties of the Markov chain.

K is used in the properties of mixing of Markov chains (expected time to stationarity)

K is used in bounding overall differences in the stationary probs of a MC subjected to perturbations.

Expressions for *K*

If $G = [g_{ij}]$ is any g-inverse of $I - P$, then

$$K = 1 + tr(G) - tr(G\Pi) = 1 + \sum_{j=1}^{m} (g_{jj} - g_{j.}\pi_j),$$

$$K = 1 - (1/m) + tr(H) = 1 - (1/m) + \sum_{j=1}^{m} h_{jj},$$

$$K = tr(Z) = \sum_{j=1}^{m} z_{jj}.$$

For any irreducible m-state MC, $K \geq \dfrac{m+1}{2}$,

$$\Rightarrow tr(H) = \sum_{j=1}^{m} h_{jj} \geq \frac{m-1}{2} + \frac{1}{m}.$$

Doubly stochastic MCs

If $c_i = 1$ for all i, then

$$m_{i\bullet} = mK = m - 1 + mtr(H) = mtr(Z).$$

Further, for all i,

$$m_{i\bullet} \geq \frac{m(m+1)}{2},$$

– a new result.

Example – Two state MCs

Let $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix} = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$,

$(0 \le a \le 1,\ 0 \le b \le 1)$. Let $d = 1 - a - b$.

MC irreducible $\Leftrightarrow -1 \le d < 1$.

MC has a unique stationary probability vector

$\boldsymbol{\pi}^T = (\pi_1, \pi_2) = (b/(a+b), a/(a+b)) = (b/(1-d), a/(1-d))$.

$-1 < d < 1 \Leftrightarrow$ MC is regular and the stationary distribution is the limiting distribution of the MC.

$d = -1 \Leftrightarrow$ MC is irreducible periodic, period 2.

Example – Two state MCs

$$\boldsymbol{c}^T = (c_1, c_2) = (1-a+b, 1+a-b).$$

a and *b* specify all the transition probabilities.

c_1 and c_2 do not uniquely specify the transition probabilities since $c_1 + c_2 = 2$.

We cannot solve for *a* and *b* in terms of c_1 and c_2

Note $c_2 - c_1 = 2(a-b)$.

Example – Two state MCs

$$H = [I - P + \mathbf{ec}^T]^{-1} = \frac{1}{2(a+b)} \begin{bmatrix} 1+a & -(1-b) \\ -(1-a) & 1+b \end{bmatrix}.$$

$$Z = [I - P + \mathbf{e}\pi^T]^{-1} = \frac{1}{a+b} \begin{bmatrix} b + \dfrac{a}{a+b} & a - \dfrac{a}{a+b} \\ b - \dfrac{b}{a+b} & a + \dfrac{b}{a+b} \end{bmatrix}.$$

$$M = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \begin{bmatrix} (a+b)/b & 1/a \\ 1/b & (a+b)/a \end{bmatrix}.$$

Example – Two state MCs

$$K = 1 + \frac{1}{a+b} \geq 1.5,$$

$$K = 1.5 \;\Leftrightarrow\; a = b = 1.$$

$$c_1 \leq c_2 \Leftrightarrow b \leq a \Leftrightarrow \pi_1 \leq \pi_2 \Leftrightarrow m_{22} \leq m_{11}$$

and

$$h_{11} \leq h_{22} \Leftrightarrow a \leq b \Leftrightarrow m_{11} + m_{21} = m_{\bullet 1} \leq m_{\bullet 2} = m_{12} + m_{22}.$$

Example – Three state MCs

$$P = \left[p_{ij}\right] = \begin{bmatrix} 1-p_2-p_3 & p_2 & p_3 \\ q_1 & 1-q_1-q_3 & q_3 \\ r_1 & r_2 & 1-r_1-r_2 \end{bmatrix}.$$

Six constrained parameters with

$0 < p_2 + p_3 \le 1,\ 0 < q_1 + q_3 \le 1$ and $0 < r_1 + r_2 \le 1.$

Let $\Delta_1 \equiv q_3 r_1 + q_1 r_2 + q_1 r_1,\ \Delta_2 \equiv r_1 p_2 + r_2 p_3 + r_2 p_2,$

$\Delta_3 \equiv p_2 q_3 + p_3 q_1 + p_3 q_3,\ \Delta \equiv \Delta_1 + \Delta_2 + \Delta_3.$

Example – Three state MCs

MC is irreducible

(and hence a stationary distribution exists)

$\Leftrightarrow \quad \Delta_1 > 0,\ \Delta_2 > 0,\ \Delta_3 > 0.$

Stationary distribution given by

$$(\pi_1, \pi_2,\ \pi_3) = \frac{1}{\Delta}(\Delta_1, \Delta_2, \Delta_3).$$

Example – Three state MCs

Let $\tau_{12} = p_3 + r_1 + r_2,\ \tau_{13} = p_2 + q_1 + q_3, \tau_{21} = q_3 + r_1 + r_2,$

$\tau_{23} = q_1 + p_2 + p_3,\ \tau_{31} = r_2 + q_1 + q_3,\ \tau_{32} = r_1 + p_2 + p_3,$

Let $\tau = p_2 + p_3 + q_1 + q_3 + r_1 + r_2$

$\Rightarrow \tau = \tau_{12} + \tau_{13} = \tau_{21} + \tau_{23} = \tau_{31} + \tau_{32}.$

$$M = \begin{bmatrix} \Delta/\Delta_1 & \tau_{12}/\Delta_2 & \tau_{13}/\Delta_3 \\ \tau_{21}/\Delta_1 & \Delta/\Delta_2 & \tau_{23}/\Delta_3 \\ \tau_{31}/\Delta_1 & \tau_{32}/\Delta_2 & \Delta/\Delta_3 \end{bmatrix}$$

Example – Three state MCs

$$H = \frac{1}{3\Delta}\begin{bmatrix} \Delta_1 & \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_2 & \Delta_3 \\ \Delta_1 & \Delta_2 & \Delta_3 \end{bmatrix}$$

$$+\frac{1}{3\Delta}\begin{bmatrix} c_2\tau_{21} + c_3\tau_{31} & -c_2\tau_{12} + c_3(\tau_{13} - \tau_{31}) & c_2(\tau_{12} - \tau_{21}) - c_3\tau_{13} \\ -c_1\tau_{21} + c_3(\tau_{23} - \tau_{32}) & c_1\tau_{12} + c_3\tau_{32} & c_1(\tau_{21} - \tau_{12}) - c_3\tau_{23} \\ -c_1\tau_{31} + c_2(\tau_{32} - \tau_{23}) & c_1(\tau_{31} - \tau_{13}) - c_2\tau_{32} & c_1\tau_{13} + c_2\tau_{23} \end{bmatrix}$$

Example – Three state MCs

Kemeny's constant:

$$K = 1 + \frac{\tau}{\Delta}.$$

For all three-state irreducible MCs, $K \geq 2$.

$K = 2$ achieved in "the minimal period 3" case

when $p_2 = q_3 = r_1$.

Example – Three state MCs

Under the imposition of column totals with $c_1 + c_2 + c_3 = 3$, we can reduce the free parameters to p_2, p_3, q_1, q_3, c_1 and c_2 by taking

$r_1 = c_1 - 1 + p_2 + p_3 - q_1,\ \ r_2 = c_2 - 1 - p_2 + q_1 + q_3.$

Let $\alpha_1 \equiv\ q_1 + q_3 - p_2,\ \alpha_2 \equiv\ p_2 + p_3 - q_1$, then

$$\pi_1 \le \pi_2 \Leftrightarrow m_{22} \le m_{11} \Leftrightarrow \Delta_1 \le \Delta_2 \Leftrightarrow r_1\alpha_1 \le r_2\alpha_2,$$

$$c_1 \le c_2 \quad \Leftrightarrow \quad r_1 + \alpha_1 \le\ r_2 + \alpha_2.$$

No universal inequalities connecting $c_1 \le c_2$ with $\pi_1 \le \pi_2$.

Example – Three state MCs

The following table gives parameter regions where the stated inequalities occur, in the case where $r_1 > 0$.

	$c_1 \le c_2$	$c_2 \le c_1$	
$\pi_1 \le \pi_2$	$\alpha_1 \le \min\left(\frac{r_2\alpha_2}{r_1}, r_2 - r_1 + \alpha_2\right)$	$r_2 - r_1 + \alpha_2 \le \alpha_1 \le \frac{r_2\alpha_2}{r_1}$	$r_1\alpha_1 \le r_2\alpha_2$
$\pi_2 \le \pi_1$	$\frac{r_2\alpha_2}{r_1} \le \alpha_1 \le r_2 - r_1 + \alpha_2$	$\max\left(\frac{r_2\alpha_2}{r_1}, r_2 - r_1 + \alpha_2\right) \le \alpha_1$	$r_2\alpha_2 \le r_1\alpha_1$
	$r_1 + \alpha_1 \le r_2 + \alpha_2$	$r_2 + \alpha_2 \le r_1 + \alpha_1$	

Example – Five state MC

Five state irreducible MC from Kemeny and Snell [10] (p199).

Rearrange the states so that the column sums are ordered with

$c_1 \ge c_2 \ge c_3 \ge c_4 \ge c_5$ with transition matrix

$$P = \begin{bmatrix} 0.759 & 0.082 & 0.065 & 0.071 & 0.023 \\ 0.095 & 0.831 & 0.033 & 0.028 & 0.013 \\ 0.112 & 0.046 & 0.788 & 0.038 & 0.016 \\ 0.156 & 0.054 & 0.045 & 0.728 & 0.017 \\ 0.107 & 0.038 & 0.034 & 0.036 & 0.785 \end{bmatrix}.$$

$$\boldsymbol{c}^T = (c_1, c_2, c_3, c_4, c_5) = (1.229,\ 1.051,\ 0.965,\ 0.910,\ 0.854)$$

$$\boldsymbol{\pi}^T = (\pi_1, \pi_2, \pi_3, \pi_4, \pi_5) = (0.3216,\ 0.2705,\ 0.1842,\ 0.1476,\ 0.0761)$$

$\Rightarrow \pi_1 \ge \pi_2 \ge \pi_3 \ge \pi_4 \ge \pi_5 !$ Not what we expected!

Example – Five state MC

$$H = \begin{bmatrix} 2.1984 & -0.5537 & -0.4911 & -0.3007 & -0.6530 \\ -0.8883 & 3.5691 & -0.9174 & -0.7613 & -0.8021 \\ -0.6457 & -1.0375 & 3.2047 & -0.5873 & -0.7342 \\ -0.2485 & -0.8505 & -0.6652 & 2.6746 & -0.7104 \\ -0.7023 & -1.2092 & -0.8680 & -0.6157 & 3.5952 \end{bmatrix}$$

All the diagonal elements of H, h_{jj}, are positive (as expected)

All the off-diagonal terms are negative.

Each row sum is 0.200.

The column sums are given as

$\boldsymbol{h}^T_{colsum} = (\ -0.2863,\ \ -0.0818,\ \ 0.2631,\ 0.4096,\ \ 0.6955)$

also ordered according to the order in $\boldsymbol{c}^T$.

Example – Five state MC

$$M = \begin{bmatrix} 3.1097 & 15.2435 & 20.0601 & 20.1581 & 55.7987 \\ 9.5987 & 3.6974 & 22.3742 & 23.2789 & 57.7567 \\ 8.8444 & 17.0326 & 5.4278 & 22.1001 & 56.8645 \\ 7.6091 & 16.3412 & 21.0051 & 6.7752 & 56.5528 \\ 9.0204 & 17.6672 & 22.1062 & 22.2926 & 13.1345 \end{bmatrix}$$

The vector of row sums of M is $(m_{\bullet 1}, m_{\bullet 2}, m_{\bullet 3}, m_{\bullet 4}, m_{\bullet 5}) =$ (114.3702, 116.7059, 110.2695, 108.2834, 84.2210), leads to no ordered relationships.

The vector of column sums is $(m_{1\bullet}, m_{2\bullet}, m_{3\bullet}, m_{4\bullet}, m_{5\bullet}) =$ (38.1824, 69.9820, 90.9734 , 94.6048, 240.1074), with $c_i \geq c_j \Rightarrow m_{i\bullet} \leq m_{j\bullet}$ for $i \leq j$.

No general results of such a nature for general finite MCs.

Example – Eight state MC

$$P = \begin{bmatrix} 0.478 & 0.270 & 0 & 0 & 0.150 & 0 & 0.055 & 0.047 \\ 0.130 & 0.870 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0.320 & 0 & 0.669 & 0.011 & 0 & 0 & 0 & 0 \\ 0.088 & 0 & 0 & 0.912 & 0 & 0 & 0 & 0 \\ 0.150 & 0 & 0 & 0 & 0.740 & 0.110 & 0 & 0 \\ 0.300 & 0 & 0 & 0.011 & 0 & 0.689 & 0 & 0 \\ 0.260 & 0 & 0 & 0 & 0 & 0 & 0.740 & 0 \\ 0.600 & 0 & 0.400 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

(Funderlic and Meyer, example involving the analysis of radio-phosphorous kinetics in an aquarium system.)

States reordered so that P has column sums with $c_i > c_j$ for $i < j$.

$\mathbf{c}^T = (2.326,\ 1.140,\ 1.069,\ 0.934,\ 0.890,\ 0.799,\ 0.795,\ 0.047)$

Example – Eight state MC

$\boldsymbol{c}^T = (2.326,\ 1.140,\ 1.069,\ 0.934,\ 0.890,\ 0.799,\ 0.795,\ 0.047)$

$\boldsymbol{\pi}^T = (0.2378,\ 0.4938,\ 0.0135,\ 0.0078,\ 0.1372,\ 0.0485,\ 0.0503,\ 0.0112).$

Note, for example $\pi_1 < \pi_2$ even though $c_1 > c_2$.

$$H = \begin{bmatrix} 1.036 & 1.056 & -0.352 & -1.403 & 0.170 & -0.261 & -0.163 & 0.043 \\ -0.792 & 4.949 & -0.456 & -1.463 & -0.885 & -0.634 & -0.550 & -0.043 \\ 0.228 & -0.623 & 2.623 & -1.052 & -0.296 & -0.426 & -0.334 & 0.005 \\ -1.666 & -4.556 & -0.505 & 9.872 & -1.389 & -0.812 & -0.735 & -0.084 \\ -0.242 & -1.599 & -0.425 & -1.275 & 3.279 & 0.839 & -0.434 & -0.017 \\ 0.176 & -0.731 & -0.401 & -1.029 & -0.326 & 2.779 & -0.345 & 0.002 \\ 0.122 & -0.844 & -0.404 & -1.433 & -0.358 & -0.448 & 3.490 & -0.000 \\ 0.475 & -0.110 & 0.825 & -1.271 & -0.154 & -0.376 & -0.282 & 1.017 \end{bmatrix}$$

Diagonal elements of H are positive. No obvious pattern for off-diagonal elemnts.

Rows sums of H = 0.125, (=1/8). Column sums of H do not exhibit any pattern

Example – Eight state MC

$$M = \begin{bmatrix} 4.21 & 7.88 & 220.32 & 1454.40 & 22.66 & 62.66 & 72.62 & 87.13 \\ 7.69 & 2.03 & 228.01 & 1462.09 & 30.35 & 70.35 & 80.32 & 94.82 \\ 3.40 & 11.28 & 74.05 & 1409.09 & 26.06 & 66.06 & 76.02 & 90.53 \\ 11.36 & 19.25 & 231.68 & 128.99 & 34.03 & 74.02 & 83.99 & 98.49 \\ 5.38 & 13.26 & 225.69 & 1437.84 & 7.29 & 39.99 & 78.00 & 92.50 \\ 3.62 & 11.50 & 223.93 & 1406.17 & 26.23 & 20.61 & 76.24 & 90.74 \\ 3.85 & 11.73 & 224.16 & 1458.24 & 26.51 & 66.50 & 19.88 & 90.97 \\ 2.36 & 10.24 & 133.19 & 1437.27 & 25.02 & 65.02 & 74.98 & 89.49 \end{bmatrix}$$

No ordered relationship within the row sums of *M*

No ordered relationship within the column sums of *M*

Kemeny's constant = 29.9194.

Conjectures:

$c_i \le c_j$ for all $i, j \Leftrightarrow \pi_i \le \pi_j$ for all $i, j \Leftrightarrow h_{ij} < 0$ for all $i \ne j$

Valid in the two-state case and the special 5-state case.

Not true in general. Example:

If $P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ then $H = \frac{2}{9}\begin{bmatrix} 3 & -1/2 & -1 \\ -3/2 & 5/2 & 1/2 \\ -3/2 & -1/2 & 7/2 \end{bmatrix}$.

$\mathbf{c}^T = (3/2, 1, 1/2)$, $\boldsymbol{\pi}^T = (1/2, 1/3, 1/6)$, $h_{23} > 0$. (Kirkland)

Are there any general inter-relationships?

If $h_{ij} < 0$ for all $i \ne j$, does $\pi_i \le \pi_j$ for all i, j?

does $c_i \le c_j$ for all i, j?

References - 1

[1] R. Funderlic, C.D. Meyer, Sensitivity of the stationary distribution vector for an ergodic Markov chain, Linear Algebra Appl. 76 (1986) 1-17.

[2] J. J. Hunter, Generalized inverses and their application to applied probability problems, Linear Algebra Appl. 45 (1982) 157-198.

[3] J. J. Hunter, Mathematical Techniques of Applied Probability, Volume 1, Discrete Time Models: Basic Theory, Academic, New York, 1983.

[4] J. J. Hunter, Mathematical Techniques of Applied Probability, Volume 2, Discrete Time Models: Techniques and Applications, Academic, New York,1983.

[5] J. J. Hunter, Parametric Forms for Generalized inverses of Markovian Kernels and their Applications, Linear Algebra Appl. 127 (1990) 71-84.

[6] J. J. Hunter, Mixing times with applications to perturbed Markov chains, Linear Algebra Appl. 417 (2006) 108-123.

[7] J. J. Hunter, Simple procedures for finding mean first passage times in Markov chains, Asia-Pacific J of Operational Research, 24 (6), (2007), 813-829.

References - 2

[8] J. J. Hunter, Variances of First Passage Times in a Markov chain with applications to Mixing Times, Linear Algebra Appl. 429 (2008) 1135-1162.

[9] J. J. Hunter, Some stochastic properties of“semi-magic” and “magic” Markov chains, Linear Algebra Appl. 433 (2010), 893-907.

[10] J. G. Kemeny, J.L. Snell, Finite Markov Chains, Van Nostrand, New York, 1960.

[11] S. Kirkland, Subdominant eigenvalues for stochastic matrices with given column sums, Electronic Journal of Linear Algebra, 18 (2009), 784-800, ISSN 1081-3810.

[12] S. Kirkland, Column sums and the conditioning of stationary distribution for a stochastic matrix, Operators and Matrices, 4 (2010), 431-443, ISSN 1846-3886.

[13] C. D. Meyer Jr., The role of the group generalized inverse in the theory of finite Markov chains, SIAM Rev.17 (1975) 443-464.